Morse Theory for Orbifolds

by

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Abstract

In Chapter 1 of this thesis, we develop the foundations of Morse theory. Morse theory, in its current incantations, has paved the way for modern developments in Floer theory, namely through the study of the geometry of manifolds using information about their smooth functions and the critical points on such functions. By doing so, one can construct the Morse complex, whose homology is isomorphic to the singular homology of the manifold.

An object of more recent interest in mathematics is the orbifold, a generalization of manifolds, which encode certain symmetries which appear naturally in the study of moduli spaces and string theory. In a paper of Cho and Hong ([CH14]), tools from Morse theory are extended to orbifolds, providing an adjusted Morse complex whose homology again agrees with the simplicial homology of the orbifold. Chapter 2 serves as a survey of this paper. Further, the density of Morse-Smale functions is discussed both for manifolds and for orbifolds, and a counterexample in the case of the latter is demonstrated. We provide two questions to be probed for future research.

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Symbols and Notation

∂M Boundary of M

(γ_1, γ_2) Broken trajectory along γ_1 then γ_2

$\mathrm{crit}(f)$ Critical points of the function f

$\mathrm{crit}_k(f)$ Critical points of the function f of index k

∂ Differential boundary map

$\bullet^G$ G-invariant part of

$Fr(X)$ Frame bundle of X

$[M/G]$ Global quotient orbifold

$C^k_G(M)$ G-invariant C^k functions

$\mathrm{ind}(p)$ Index of a critical point p

G_p Isotropy group at p

M (Riemannian) manifold

$\mathcal{L}(p, q)$ Moduli space of flow trajectories from p to q

$CM_*(X, f; R)$ Morse complex of X defined by f, with coefficients in R

$HM_k(M)$ kth Morse homology group of M

$n(p, q)$ Number of flow trajectories from p to q

$\mathbf{X}$ Orbifold

$(\tilde{U}, G, \pi)$ Orbifold (uniformizing) chart

$\mathrm{crit}^{\pm}(f)$ Orientable (+)/non-orientable (−) critical points of the function f

X pseudo-gradient vector field, or underlying quotient space

Θ_p^- Orientation space at p

$H_k(M)$ kth singular homology group of M

$W^{\pm}(p)$ Stable (+)/unstable (−) manifold of p

M^a Sublevel set $f^{-1}((-\infty, a])$

Chapter 1

Morse Theory for Manifolds

1.1 Foundations of Morse Theory

Morse theory, developed by Marston Morse in the first half of the 20th century, is a technique in differential topology which allows one to extract topological information about a manifold M by studying smooth functions $f : M \to \mathbb{R}$ whose critical points are non-degenerate (i.e., its Hessian is non-singular). One of the motivations of studying these so-called *Morse* functions $f : M \to \mathbb{R}$ on some manifold M is to understand the topology of the sublevel surfaces $M^a := f^{-1}((-\infty, a]) = \{p \in M : f(p) \leq a\}$.

It turns out that via the following lemma, one can completely determine the local behavior of a Morse function f by studying the *index* of its critical points. Consider the Hessian of a Morse function f at a critical point p as a symmetric bilinear functional on the vector space T_pM. The index of this functional (i.e., the maximal dimension of a subspace of T_pM on which the Hessian is negative definite), also referred to as the index of the critical point p (of f).

Lemma 1.1.1 (Morse)**.** Let p be a nondegenerate critical point of $f : V \to \mathbb{R}$. Then there exists a neighborhood $U \ni p$ and a diffeomorphism $\varphi : (U, p) \to (\mathbb{R}^n, 0)$ such that

$$f \circ \varphi^{-1}(x_1, ..., x_n) = f(p) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

Remark 1.1.1. A chart in whose open set the coordinates given by the Morse Lemma are defined is called a *Morse chart*.

One can recall that a function is closely approximated in a neighborhood by a quadratic function which is associated to the function's second derivative. The Morse Lemma builds upon this, giving that after a possible change of coordinates, the two are indeed equal. We can use the Morse lemma to prove the following corollary.

Corollary 1.1.1. Critical points of a Morse function are isolated.

Proof. Suppose $p \in \text{crit}(f)$, such that $f(x) = p$. The Morse lemma gives that there exists some diffeomorphism φ such that

$$(f \circ \varphi^{-1})(x_1, ..., x_n) = p - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

In this change of coordinates, we have that $\frac{\partial(f \circ \varphi^{-1})}{\partial x_i} = \pm 2x_i$, and so the Jacobian is 0 exactly when $x_i = 0$ for all i. This only occurs when $\varphi(x) = 0$. $\square$

This lemma also inspires the two "foundational theorems of Morse theory," which allow one to classify the homotopy type of the sublevel sets M^a purely by analyzing the behavior of critical points of a Morse function, are as follows:

Theorem 1.1.1 (Foundational Theorem 1, [Mil69])**.** Let M be a manifold, and $f : M \to \mathbb{R}$ be Morse. Let $a < b$, and suppose that the set $f^{-1}([a, b]) := \{x \in M : a \leq f(x) \leq b\}$ is compact and contains no critical points of f. Then the set M^a is diffeomorphic to M^b; further, M^a is a deformation retract of M^b, so that the inclusion map $M^a \to M^b$ is a homotopy equivalence.

If ψ denotes the flow of the gradient vector field for some metric on M, a suitable deformation retract from M^b onto M^a as indicated in the above theorem would be the map

$$r : M^b \times [0, 1] \to M^b, \qquad (x, s) \mapsto \begin{cases} x & \text{if } f(x) \leq a, \\ \psi^{s(f(x)-a)}(x) & \text{if } a \leq f(x) \leq b. \end{cases}$$

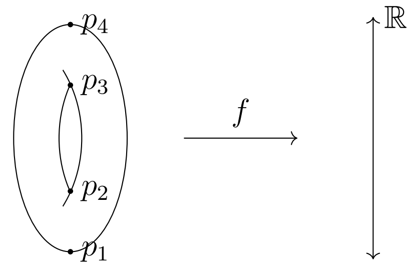

Figure 1.1: *The height function on the torus*

Theorem 1.1.2 (Foundational Theorem 2, [Mil69])**.** Let M be a manifold, and $f : M \to \mathbb{R}$ be Morse. Let p be a non-degenerate critical point with index λ. If we define $f(p) = c$, suppose that $f^{-1}([c - \varepsilon, c + \varepsilon])$ is compact and $\varepsilon > 0$ is such that the region contains no critical points of f other than p. Then for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-handle attached, where λ denotes the index of p.

Effectively, these theorems (the first of which will be proven using more advanced techniques in the next section) describe how we can decompose a manifold into components (handle decomposition) which are entirely determined by the behavior of the critical points of a suitable function. Indeed, the first developments of Morse theory prove to be a highly useful tool, notably being used in Smale's celebrated proof of the h-cobordism theorem. A visual example of these behaviors follows.

Example 1.1.1. Consider the torus T^2, oriented so that it is standing "vertically" as shown in Figure 1.1. In order to apply the theorems above, we need a Morse function $f : T^2 \to \mathbb{R}$, which we can write by considering T^2's embedding into $\mathbb{R}^3$. In particular, we write $f : T^2 \subset \mathbb{R}^3 \to \mathbb{R}$ defined by the projection $(x, y, z) \mapsto z$. When the torus is oriented as shown, there are four critical points. Notice that the indices of these points are 0, 1, 1, and 2, from bottom to top. We explicitly apply Theorem 1.1.1 to describe how different sublevel sets are diffeomorphic when taken between the same critical points.

For notation, we write $R_i = f^{-1}[f(p_{i-1}) + \varepsilon_1^i, f(p_i) - \varepsilon_2^i]$ with $\varepsilon_n^i > 0$ to denote each of these regions. By definition, there are no critical points contained in any R_i.

R_1: Consider the points $a_1 = f(p_1) - \varepsilon^1$ and $b_1 = f(p_1) - \frac{\varepsilon^1}{2}$. Then $M_1^a = f^{-1}(-\infty, a_1] = \emptyset$ and $M_1^b = f^{-1}(-\infty, b_1] = \emptyset$. Trivially, the theorem gives that $\emptyset \cong \emptyset$ and $\emptyset \simeq \emptyset$.

R_2: Consider the points $a_2 = f(p_1) + \varepsilon_1^2$ and $b_2 = f(p_2) - \varepsilon_2^2$. Then we have $M_2^a = f^{-1}(-\infty, a_2]$ and $M_2^b = f^{-1}(-\infty, b_2]$. The theorem gives that $M_2^a \cong M_2^b$, as well as $M_2^a \simeq M_2^b$, so we can deform either M_2^a or M_2^b to the other.

R_3: Identical treatment as R_2, so $M_3^a \cong M_3^b$ and $M_3^a \simeq M_3^b$, so we can deform either M_3^a or M_3^b to the other.

R_4: Identical treatment as R_2, so $M_4^a \cong M_4^b$ and $M_4^a \simeq M_4^b$, so we can deform either M_4^a or M_4^b to the other.

Now, we apply Theorem 1.1.2 to the same torus, in order to display the handle decomposition induced by the order of the critical points of f. Examining each critical point of f, we have the following:

p_1: Recall $\lambda_1 = 0$. Therefore, the homotopy type of the set $M^{f(p_1)+\varepsilon_1} = f^{-1}(-\infty, f(p_1)+\varepsilon_1]$ is the homotopy type of $\emptyset$, with a 0-cell (a point) attached. Note that a point is homotopy equivalent to a disk.

p_2: Recall $\lambda_2 = 1$. Therefore, the homotopy type of the set $M^{f(p_2)+\varepsilon_2} = f^{-1}(-\infty, f(p_2)+\varepsilon_2]$ is of the homotopy type of $M^{f(p_1)+\varepsilon_1}$, with an additional 1-handle attached. We can think of this as a cylinder.

p_3: Recall $\lambda_3 = 1$. Therefore, the homotopy type of the set $M^{f(p_3)+\varepsilon_3} = f^{-1}(-\infty, f(p_3)+\varepsilon_2]$ is of the homotopy type of $M^{f(p_2)+\varepsilon_2}$, with an additional 1-handle attached. We can think of this as a torus with a disk removed.

p_4: Recall $\lambda_4 = 2$. Therefore, the homotopy type of the set $M^{f(p_4)+\varepsilon_4} = f^{-1}(-\infty, f(p_4)+\varepsilon_4]$ is of the homotopy type of $M^{f(p_3)+\varepsilon_3}$, with an additional 2-handle attached. A 2-handle is a disk, and so this is indeed the full torus.

Indeed, we see that the torus is successfully handle-decomposed by using information about the critical points of an arbitrarily chosen Morse function. ◇

A notable result whose proof is highly dependent on these theorems of Morse is the Reeb sphere theorem, as follows:

Theorem 1.1.3 (Reeb)**.** If M is a compact manifold, and if there exists some Morse function on M with only two critical points, then M is homeomorphic to S^2.

Proof. Geometrically, we must have that the two critical points are the maximum and minimum of M. Without loss of generality, we can assume $f(M) = [0, 1]$. Now for sufficiently small $0 < \varepsilon < 1$, the Morse lemma gives that $f^{-1}([0, \varepsilon])$, $f^{-1}([1 - \varepsilon, 1])$ are disks, D^n. The first Morse theorem above gives that the sets M^ε and $M^{1-\varepsilon}$ are diffeomorphic, which in turn gives that $M^{1-\varepsilon}$ is also a disk. Now we have that M is the union of two disks glued along ∂D^n. Certainly, this is S^2. □

Remark 1.1.2. Note that the technique used in the above proof is known as Alexander's trick, however there are some nuances. Namely, this fails in the smooth category.

1.2 Morse Theory using Gradient Flows

In more recent years since the original work of Morse, Morse theory has evolved to study geodesics. In particular, new ideas have been built off of the idea of pseudo-gradient flows which can be paired with Morse functions on Riemannian manifolds. By doing so, we can form a strong definition of negative (pseudo)-gradient flow lines between critical points. This is done by considering the stable and unstable manifolds for each critical point, that is, sets which act as attractors or repellors along gradient flows.

Definition 1.2.1. A *pseudo-gradient vector field* X of a Morse function $f : M \to \mathbb{R}$ is a vector field on a Riemannian manifold M such that $df(X) < 0$ for all X except at critical points (at which it is 0, by definition), and near any critical point there exists a Morse chart (recall Remark 1.1.1) such that X is the gradient of f for the Euclidean metric on the chart.

We find that pseudo-gradient fields provide a nice alternative to gradient fields, as they have the same behavior while removing dependence on an inner product structure.

Remark 1.2.1. Pseudo-gradient vector fields exist for all Morse functions on all manifolds. This is a consequence of the existence of Riemannian metrics with a prescribed form on a given subset of the manifold, particularly in a neighborhood of the critical points.

In other words, in a Morse chart $U \ni x$, the pseudo-gradient vector field X coincides with the negative gradient for the Euclidean metric. These are important, because X contains the flow lines which connect critical points of f; all trajectories go from one critical point to another. In particular, consider the following.

Proposition 1.2.1. The Morse function f is non-increasing along gradient flow lines, and is strictly decreasing along a flow line which does not contain a critical point of the function.

In order to formalize the idea of attraction and repulsion from critical points, we can introduce the notion of stable and unstable manifolds. These objects will allow us to later define a moduli space of flow trajectories, which can be used to construct Morse homology.

Definition 1.2.2. The *stable manifold* $W^+(p)$ of $p \in \text{crit}(f)$ is the point itself along with all regular points whose integral lines of a pseudo-gradient end at p, that is all points whose trajectory along gradient vector field leads to p. Similarly, *unstable manifold* $W^-(p)$ of $p \in \text{crit}(f)$ is the set of points near p whose trajectories along the pseudo-gradient vector field diverge from p. Precisely, for M a manifold and φ^s the flow of a pseudo-gradient field,

$$W^+(p) = \{x \in M : \lim_{s \to \infty} \varphi^s(x) = p\}, \qquad W^-(p) = \{x \in V : \lim_{s \to -\infty} \varphi^s(x) = p\}.$$

Notice that since we are adapting these stable/unstable manifolds to a *negative* pseudo-gradient flow as prescribed by Morse theory, we have that $f(p) \geq f(x)\ \forall x \in W^+(u)$ and $f(p) \leq f(x)\ \forall x \in W^-(p)$.

Proposition 1.2.2. If p is a critical point, then $W^+(p),\ W^-(p)$ are submanifolds of V which are diffeomorphic to an open disk. Moreover, $\dim W^-(p) = \operatorname{codim} W^+(p) = \operatorname{ind}(p)$.

Each critical point of a Morse function f lives on some flow line. Since the unstable manifolds of distinct critical points are disjoint, we can decompose M in a way similar to a CW-complex.

The purpose of considering the flows of a pseudo-gradient vector field X comes from the fact that these flows always connect critical points of f. In fact, let us use these ideas to finally prove Theorem 1.1.1, the first Foundational Theorem of Morse Theory.

Proof of Theorem 1.1.1. We want to use the flow of X to retract M^b to M^a. Fix a function $\rho : M \to \mathbb{R}$ with values

$$\begin{cases} -((df)_x(X))^{-1} & \text{on } f^{-1}([a,b]), \\ 0 & \text{outside of a compact neighborhood of } f^{-1}([a,b]). \end{cases}$$

The vector field $Y = \rho X$ is equivalent to 0 outside of a compact set, so that its flow φ^s is defined for all $s \in \mathbb{R}$. For a fixed $x \in M$, consider the function $s \mapsto f \circ \varphi^s(x)$. If $\varphi^s(x) \in f^{-1}([a,b])$, then we have

$$\begin{aligned} \frac{d}{ds} f \circ \varphi^s(x) &= (df)_{\varphi^s(x)} \left(\frac{d}{ds} \varphi^s(x) \right) \\ &= (df)_{\varphi^s(x)}(Y_{\varphi^s(x)}) \\ &= -1. \end{aligned}$$

Therefore, for $\varphi^s(x) \in f^{-1}([a,b])$, we have $f \circ \varphi^s(x) = -s + f(x)$, and so the diffeomorphism (flow) φ^{b-a} of M sends M^b onto M^a. □

A powerful reason for studying Morse theory using gradient flows is that it allows for the construction of a moduli space which one can use to form a chain complex whose homology is isomorphic to the singular homology of the manifold in question. This is explored in depth in the next section.

1.3 The Morse Complex

For Morse functions which have stable and unstable manifolds which satisfy certain transversality requirements, called *Morse-Smale* functions, we are able to construct the *Morse complex*. In particular, we must have the following condition:

Definition 1.3.1. The *Smale (transversality) condition* for the pair (f, X), where f is a Morse function and X a pseudo-gradient vector field adapted to f on the manifold M, is

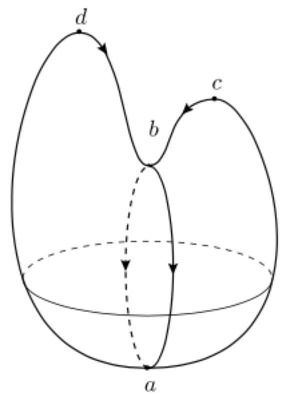

Figure 1.2: S^2 *with a "dent" in the top, creating four critical points. Image from [AD13].*

satisfied if for every pair of critical points $p, q \in \text{crit}(f)$, we have that $W^-(p) \pitchfork W^+(q)$.

Example 1.3.1. Certain stable and unstable manifolds always meet transversally. Trivially, for the same critical point p, we have that $W^-(p) \pitchfork W^+(p)$, and also $W^-(p) \cap W^+(q) = \emptyset$ (i.e., they are transversal) for distinct critical points p, q such that $f(p) \leq f(q)$.

Example 1.3.2. An important counterexample is the torus T^2 with the usual height function $(x, y, z) \mapsto z$ as seen in Figure 1.1. In particular, if one considers the flow trajectories between the middle two critical points of index 1, we see that there are two trajectories. This does not satisfy the Smale condition. However, if one were to tilt the torus along a non-trivial axis, this problem would be avoided.

Example 1.3.3. Consider the dented sphere in Figure 1.2. Notice that at each critical point, any two unstable/stable manifolds will always meet transversally, most notably at the saddle point.

Remark 1.3.1. The Smale condition is solely dependent on the Riemannian metric g. That is, one can not attain/lose the Smale condition purely by taking a different choice of Morse function on the manifold. However, there do always exist metrics which can be equipped which satisfy the Smale condition (consider the bumpy metric). In particular, the "Kupka-Smale theorem" provides a more detailed explanation for this idea.

It turns out that when a vector field satisfies the Smale condition, we have that for all critical points p, q that

$$\operatorname{codim}(W^-(p) \cap W^+(q)) = \operatorname{codim}(W^-(p)) + \operatorname{codim}(W^+(q)),$$

or equivalently,

$$\dim(W^-(p) \cap W^+(q)) = \operatorname{ind}(p) - \operatorname{ind}(q).$$

This intersection $W^-(p) \cap W^+(q)$ under the Smale condition turns out to be a submanifold of M. We denote this intersection, which consists of all points on the trajectories connecting p and q, by

$$\mathcal{M}(p, q) = \left\{ x \in M : \lim_{s \to -\infty} \varphi^s(x) = p, \ \lim_{s \to \infty} \varphi^s(x) = q \right\}.$$

The group $\mathbb{R}$ acts on $\mathcal{M}(p, q)$ by translation in time via $s \cdot x = \varphi^s(x)$. This action is free (i.e., stabilizers are trivial) if $p \neq q$, and therefore the quotient

$$\mathcal{L}(p, q) := \mathcal{M}(p, q)/\mathbb{R}$$

is also a manifold. It can be described as the set of all trajectory flow lines (as opposed to the individual points on each of the flow lines) connecting p and q. Note that $\dim \mathcal{L}(p, q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$.

Defining this moduli space of gradient trajectories allows us to define the Morse complex. In order to provide a digestible construction of the Morse complex, we will first present these constructions over $\mathbb{Z}/2\mathbb{Z}$, although any commutative ring would suffice. Later, we shall extend these constructions to their standard form over $\mathbb{Z}$. Throughout, consider M to be a compact manifold with a Morse function f and pseudo-gradient field X which satisfies the Smale condition.

Definition 1.3.2. $CM_k(M, f; \mathbb{Z}/2) = \{\sum_{c \in \operatorname{crit}_k(f)} a_c c : a_c \in \mathbb{Z}/2\}$, where $\operatorname{crit}_k(f)$ are the critical points of index k of f, are free $\mathbb{Z}/2\mathbb{Z}$-modules defined as formal sums of critical points of index k. When the context is clear, we can just write CM_k, or some variation of information to be included. For example, CM_k, $CM_k(f;\mathbb{Z}/2\mathbb{Z})$, $CM_*(f;\mathbb{Z}/2)$ and $CM_k(M,f)$ all have the same meaning.

In order to consider these vector spaces in the context of a chain complex, and to compute homology, we need to define the differential boundary map $\partial_X = \partial^{k+1} : CM_{k+1}(f;\mathbb{Z}/2) \to CM_k(f;\mathbb{Z}/2)$. It suffices to define $\partial_X(p_{k+1})$ for $p_{k+1} \in \text{crit}_{k+1}(f)$.

Definition 1.3.3. The differential ∂_X is a linear combination of points of index k, that is,

$$\partial_X(p_{k+1}) = \sum_{p_k \in \text{crit}_k(f)} n_X(p_{k+1}, p_k)p_k, \quad \text{with } n_X(p_{k+1}, p_k) \in \mathbb{Z}/2.$$

We define the number $n_X(p,q)$ to be the number of trajectories of X from critical points p to q, mod 2 (for the case of $\mathbb{Z}/2\mathbb{Z}$). Note that later, we will use $N_X(p,q) \in \mathbb{Z}$ instead, which denotes the signed cardinalities of $\mathcal{L}(p,q)$ as coefficients.

Theorem 1.3.1. The family $(CM_*(f;\mathbb{Z}/2))_*$ when equipped with the differential ∂_X forms a complex.

We aim to show that $(CM_*(f;\mathbb{Z}/2), \partial_X^\bullet)$ is indeed a complex, in order to be able to compute (co)homology. Recall that a complex is a family $\{CM_*\}_*$ of vector spaces connected by linear maps $\partial^\bullet : CM_* \to CM_{*-1}$, such that $\partial^k \circ \partial^{k+1} = 0$; equivalently, these differentials must form something *similar, but weaker* than an exact sequence. That is, we want the condition that $\text{im}\partial^{k+1} \subset \ker \partial^k$. This allows us to take a defined quotient $H_k = \ker \partial^k / \text{im}\partial^{k+1}$, which is defined as the k^{th} homology of the complex.

So in order to achieve this goal, we need to show that the boundary maps are well defined, that is, $\mathcal{L}(p_{k+1}, p_k)$ is finite for each p_k, p_{k+1}, and also $\partial^2 = 0$. In particular, since

$$\partial_X \circ \partial_X(p_{k+1}) = \sum_{p_{k-1} \in \text{crit}_{k-1}(f)} \left(\sum_{p_k \in \text{crit}_k(f)} n_X(p_{k+1}, p_k) n_X(p_k, p_{k-1}) \right) p_{k-1},$$

we aim to show that

$$\sum_{p_k \in \text{crit}_k(f)} n_X(p_{k+1}, p_k) n_X(p_k, p_{k-1}) = \left| \coprod_{p_k \in \text{crit}_k(f)} \mathcal{L}(p_{k+1}, p_k) \times \mathcal{L}(p_k, p_{k-1}) \right| = 0.$$

The strategy then is to appeal to some facts about 1-dimensional manifolds, namely that (compact) manifolds of dimension 1 with boundary have a boundary consisting of an even number of points. So, it suffices to show that if $\coprod_{p_k} \mathcal{L}(p_{k+1}, p_k) \times \mathcal{L}(p_k, p_{k-1})$ is a boundary of a 1-dimensional manifold, then it has an even number of points. That is, we would have that the sum is 0 mod 2, satisfying $\partial^2 = 0$. In order to prove this, however, we will need a compactification argument and the notion of a broken trajectory, both to be detailed later in the following section.

1.4 Broken Trajectories

Similar to how we can track trajectories between critical points via the moduli space $\mathcal{L}(p, q)$, we can also study *broken trajectories*, which are trajectories between sequences of critical points, which may not be globally smooth (*consider the dented sphere, and the broken trajectory from an index 2 maximum, down to the index 1 saddle, down to the index 0 minimum*). In other words, we are interested in looking at flows which take "pit-stops" along various critical points. As notation, we define (γ_1, γ_2) to be a broken trajectory from p_1 to p_3, where γ flows from p_1 to p_2, and γ_2 from p_2 to p_3.

Definition 1.4.1. We define the set of broken trajectories from critical points p to q as

$$\overline{\mathcal{L}}(p, q) = \bigcup_{r_i \in \text{crit}(f)} \mathcal{L}(p, r_1) \times \cdots \times \mathcal{L}(r_{q-1}, q),$$

where the r_i are also critical points. Again, recall that we must decrease in index along the broken trajectory, and so only the $\mathcal{L}(r_i, r_{i+1})$ where $\text{ind}(r_i) > \text{ind}(r_{i+1})$ contribute to this definition.

Our reasoning for introducing this notion is to resolve the proof of Theorem 1.3.1. Indeed, it turns out that when endowed with the product topology, $\overline{\mathcal{L}}(p,q)$ is a compactification of $\mathcal{L}(p,q)$. It turns out that the following theorem gives that the differential squares to 0:

Theorem 1.4.1. If $\operatorname{ind}(p) = \operatorname{ind}(q) + 2$, then $\overline{\mathcal{L}}(p,q)$ is a compact manifold of dimension 1 with boundary.

Proof of Theorem 1.3.1 continued. If p, q are two critical points of f, then the intersection $W^-(p) \cap W^+(q)$ (which we assume is transversal) is an oriented manifold. The same holds for its intersection with a regular level set, giving that $\mathcal{L}(p,q)$ is an oriented manifold. Since compact manifolds with boundary of dimension 1 have an even number of boundary points, we have that

$$\left| \coprod_{p_k \in \operatorname{crit}_k(f)} \mathcal{L}(p_{k+1}, p_k) \times \mathcal{L}(p_k, p_{k-1}) \right| = 0 \bmod 2$$

as desired. □

1.5 Morse Homology

The notion of a chain complex (i.e. pairs (CM_*, ∂_*)) in mathematics is an important concept which allows for the computation of homology, which encodes certain information about a topological space; for example, the dimension of the kth homology of a space is equivalent to the number of "holes" in the space in the kth dimension. In particular, the Morse complex encodes as its k-chain the $\mathbb{Z}$-module which is generated by the critical points of index k, and the differentials are defined as being a linear combination of critical points of index $k-1$, with scalars defined by the cardinality of $\mathcal{L}(p,q)$ for suitable p, q.

From this chain complex, we are able to construct a very well-behaved homology theory, aptly named *Morse homology*. Speaking to this behavior, we have the powerful theorem:

Theorem 1.5.1 ([AD13]). The Morse homology HM_* of a manifold M is isomorphic to its singular homology.

Over $\mathbb{Z}/2\mathbb{Z}$, we have the following example, as promised.

Example 1.5.1. Consider the "dented" sphere from Figure 1.2, i.e. a sphere with a "dent" on the top, so there is a minimum (a), a saddle (b), and two local maxima (c and d). Computing Morse homology over $\mathbb{Z}/2$ gives the following:

1. First, there is one 0-cell (i.e. critical point of index 0) which is a, so $CM_0 = \mathbb{Z}/2$, generated by a. Next, there is one 1-cell (i.e. critical point of index 1) which is b, so $CM_1 = \mathbb{Z}/2$, generated by b. Last, there are two 2-cells (i.e. critical points of index 2) which are c and d, so $CM_2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by c and d.

2. We now want to consider the differential maps, which send elements of $CM_2 \overset{\partial^2}{\mapsto} CM_1 \overset{\partial^1}{\mapsto} CM_0$. Notice though that each differential acts as a coefficient giving the number of trajectories between critical points. So, $\partial^2 d = b$ and $\partial^2 c = b$ (local max to saddle), and $\partial^1 b = 2a = 0 \bmod 2$ (saddle down to minimum, two possible directions).

3. Last, we aim to compute the actual homology groups of the dented sphere. Recall that $H_k = \ker \partial^k / \text{im}\partial^{k+1}$. So, we consider first $H_0 = \ker \partial^0 / \text{im}\partial^1$. Notice that trivially, if $\partial^0 : CM_0 \to CM_{-1}$, we have that $\ker \partial^0 = CM_0 = \mathbb{Z}/2$ since $CM_{-1} = 0$. Also, notice that $\partial^1 = 0$, and therefore $\text{im}\partial^1 = 0 = \text{id}_{\mathbb{Z}/2}$. Certainly then, $H_0 = (\mathbb{Z}/2)/\text{id}_{\mathbb{Z}/2} = \mathbb{Z}/2$. Next, we compute $H_1 = \ker \partial^1 / \text{im}\partial^2$. Notice that ∂^1 is the zero map, and therefore $\ker \partial^1 = \mathbb{Z}/2$. Also, $\text{im}\partial^2 = \mathbb{Z}/2$ since $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$ is not just the zero map. Therefore $H_1 = (\mathbb{Z}/2)/(\mathbb{Z}/2) = 0$. Lastly, we compute $H_2 = \ker \partial^2 / \text{im}\partial^3$. Notice that ∂^2 is not injective. If we denote $c = (1, 0) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $d = (0, 1) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$, we know that the elements which map to 0 must be $(0, 0)$ and $(1, 1)$, which we can identify as $\mathbb{Z}/2$. Finally $\text{im}\partial^3 = 0$ since it is a homomorphism, and $0 \mapsto 0$. Therefore $H_2 = (\mathbb{Z}/2)/0 = \mathbb{Z}/2$.

Homology over $\mathbb{Z}$ is similar. We define $CM_k(f; \mathbb{Z}) = \{\sum_{c \in \text{crit}_k(f)} a_c c : a_c \in \mathbb{Z}\}$, in other words it is the free $\mathbb{Z}$-module generated by the critical points of f, i.e. $CM_k(f; \mathbb{Z}) = \mathbb{Z}\text{crit}_k(f)$.

In order to define the differential map $\partial_X : CM_k(f) \to CM_{k-1}(f)$, notice that we are going from a critical point of order k to a critical point of order $k-1$. We know that when $\text{ind}(p) - \text{ind}(q) = 1$, that $\mathcal{L}(p,q)$ is an oriented compact manifold of dimension 0. Therefore, it is a finite collection of points, each of which is endowed with a sign which comes from the orientation of the manifold. Denote by $N_X(p,q) \in \mathbb{Z}$ the sum of these signs. Then we can define the differential map ∂_X by $p \mapsto \sum_{q \in \text{crit}_{k-1}(f)} N_X(p,q) \cdot q = \sum_{q \in \text{crit}_{k-1}(f)} \#\mathcal{L}(p,q) \cdot q$. Note that $n_X(p,q) = N_X(p,q) \bmod 2$.

Certainly homology is a topological invariant. This is demonstrated in the following theorem, which states that homology does not depend on the choice of Morse function f nor pseudo-gradient X.

Theorem 1.5.2. Let M be a compact manifold. Let $f_0, f_1 : M \to \mathbb{R}$ be two Morse functions and let X_0 and X_1 be pseudo-gradients adapted to f_0 and f_1, respectively, with the Smale property. Then there exists a morphism of complexes $\Phi_* : (CM_*(f_0), \partial_{X_0}) \to (CM_*(f_1), \partial_{X_1})$ which induces an isomorphism in the homology.

A direct application of this theorem for the $\mathbb{Z}/2\mathbb{Z}$ case follows, with proof. This will allow us to define the Morse inequalities, which allow us to use the Betti numbers to bound the number of critical points for a Morse function.

Corollary 1.5.1. The number of critical points modulo 2 of a Morse function depends only on the manifold, and not on the function.

Proof. Consider the complex associated with a Morse function f and a pseudo-gradient X:

$$0 \xrightarrow{\partial^{n+1}} CM_n \xrightarrow{\partial^n} CM_{n-1} \longrightarrow \cdots \xrightarrow{\partial^1} CM_0 \xrightarrow{\partial^0} 0.$$

This then gives the following inequalities.

$$\begin{aligned}\#\text{crit}(f) &= \sum_{k=0}^{n} \dim CM_k(f) \\ &= \sum_{k=0}^{n+1} (\dim \ker \partial^k + \dim \text{im} \partial^k)\end{aligned}$$

$$= \sum_{k=0}^{n} (\dim \ker \partial^k + \dim \mathrm{im} \partial^{k+1})$$
$$= \sum_{k=0}^{n} (\dim \ker \partial^k - \dim \mathrm{im} \partial^{k+1}) \mod 2$$
$$= \sum_{k=0}^{n} \dim HM_k(M; \mathbb{Z}/2\mathbb{Z}) \mod 2.$$

To note, we have just shown that

$$\sum_k (-1)^k \dim CM_k = \sum_k (-1)^k \dim HM_k, \qquad (\star)$$

however we have used the rank-nullity theorem; it is then wise to consider vector spaces, i.e., to work over a field. This motivates our usage of $\mathbb{Z}/2$ to introduce these concepts.

The number $(\star)$ that we just found is the Euler characteristic of the manifold modulo 2. So, we have that $\chi(M)$ is precisely the alternating sum of the number of critical points of a Morse function on it. That is,

$$\#\mathrm{crit}(f) \geq \sum_{k=0}^{n} \dim HM_k(M; \mathbb{Z}/2\mathbb{Z}).$$

The corollary terminates with the statement of the following proposition.

Corollary 1.5.2. The number of critical points of a Morse function f on a manifold M is greater than or equal to the sum of the dimensions of the Morse homology groups (modulo 2) of M. More precisely,

$$\#\mathrm{crit}_k(f) = \dim(CM_k(M; \mathbb{Z}/2\mathbb{Z})) \geq \dim(\ker \partial_k) - \dim(\mathrm{im} \partial_{k-1}) = \dim(HM_k(M; \mathbb{Z}/2\mathbb{Z})).$$

□

Chapter 2

Morse Theory for Orbifolds

2.1 Orbifolds

Morse theory has been studied on manifolds since its inception, however there remain many open questions when one considers the case of orbifolds. Orbifolds are a generalization of manifolds, which allow for the existence of singular points. In particular, the type of orbifold which of interest is one which can be described as a manifold "quotiented" by a finite group action. A natural example would be the sphere S^2, modded out by the action of $\mathbb{Z}/n\mathbb{Z}$ along the (for example) z axis; that is, we mod out by rotations of $2\pi/n$ radians. Such an object looks, in a way, like a football, with two *orbifold points*, or *singularities*, on opposite ends along the z axis. We call this type of orbifold a "global quotient orbifold."

As mentioned, an orbifold X is like a manifold, however it is locally modeled on $\mathbb{R}/G$, with G a finite group which acts on $\mathbb{R}$. A global quotient orbifold is a particular type of orbifold, in which G does not vary across the orbifold. We write such a global quotient as $X = [M/G]$. Explicitly, we can describe orbifold charts as follows.

Definition 2.1.1. Let X be a topological space. An *orbifold chart* or *uniformizing chart* $(\tilde{U}, G, \pi)$ of dimension n for an open set $U \subset X$ consists of a connected, open subset $\tilde{U} \subset \mathbb{R}^n$, a finite group G acting smoothly and effectively (i.e., only the identity acts trivially) on $\tilde{U}$, and a continuous G-invariant map $\pi : \tilde{U} \to X$ which induces a homeomorphism $\tilde{U}/G \cong U$.

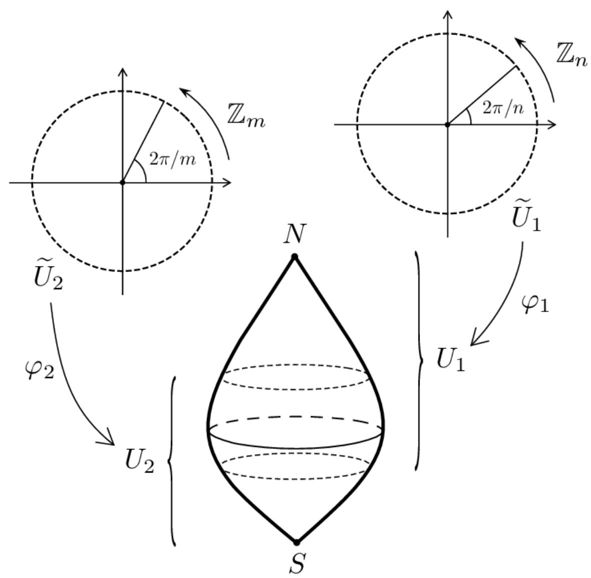

Figure 2.1: *The football orbifold. Image from [Dra11]*

Example 2.1.1. Consider the "football" orbifold in Figure 2.1, wherein a sphere is modded out by $\mathbb{Z}/n\mathbb{Z}$ in the chart φ_1, causing an orbifold point at the North pole N, and by $\mathbb{Z}/m\mathbb{Z}$ in the chart φ_2, causing an orbifold point at the South pole S. When $n = m$, this is a global quotient orbifold $[S^2/(\mathbb{Z}/n\mathbb{Z})]$, where we mod out by a $\mathbb{Z}/n\mathbb{Z}$ action in every chart, or in this picture, along the entire vertical axis. Otherwise, this is an orbifold which is not a global quotient.

Orbifolds are important objects in mathematics, notably in the study of string theory and the study of moduli spaces. Indeed, orbifolds appear in compactifications of string theory, and moduli spaces frequently take the form of orbifolds.

2.2 Morse-Smale-Witten Complex for Global Quotients

In the beginning sections of [CH14], the idea of Morse complexes is extended to global quotient orbifolds. One of the main differences when dealing with orbifolds versus manifolds is the need to consider the orientability of critical points. This is because group actions can reverse the orientation of the unstable manifolds of critical points, altering the required condition that chain complexes are such that $\partial^2 = 0$.

In this section, we will introduce two methods of generating a Morse complex (or also called a *Morse-Smale-Witten complex*) for global quotient orbifolds. First, we will consider the orientability of critical points to define subcomplexes of the "standard" Morse complex, and to consider G-actions on them. Second, we shall introduce the "orientation spaces" of critical points whose G-invariant part becomes the chain complex from the first approach.

While the second approach will track more data (in particular, the group actions studied will inherently carry information about the orientations of unstable manifolds), the first method is the only one which can be extended to general orbifolds. For this reason, we will go in to more depth for the first approach.

2.2.1 Orbifold Critical Points

Just as we defined a Morse function for a smooth manifold, we adapt this definition to orbifold charts.

Definition 2.2.1. A smooth function $\bar{f} : X \to \mathbb{R}$ is *Morse* if every point $\bar{x}$ in the orbifold has a chart $(\tilde{U}_{\bar{x}}, G_{\bar{x}}, \pi_{\bar{x}})$ where $\tilde{U} \subset \mathbb{R}^n$ is open and connected, $G \curvearrowright \tilde{U}$, and $\pi : \tilde{U} \to U \subset X$, such that $\bar{f} \circ \pi_{\bar{x}}$ is Morse (in the usual sense) on $\tilde{U}_{\bar{x}}$.

In particular, we shall study a G-invariant Morse function $f : M \to \mathbb{R}$ (which always exists), which induces a Morse function $\bar{f}$ on the global quotient orbifold $X = [M/G]$. To note, we shall assume the Morse function $f : M \to \mathbb{R}$ on a manifold to satisfy the Smale condition throughout this chapter. However, this condition becomes significantly more restrictive in the orbifold case, since the pseudo-gradient must also maintain the G-invariance of f. That is, it is possible to perturb some $f : M \to \mathbb{R}$ to make it Morse-Smale, but by doing so, we interrupt its G-invariance.

We shall now define the orientability of a critical point, and then by considering group actions on the subcomplex generated by just the orientable critical points, we will construct a Morse complex for this G-invariant Morse-Smale function f.

Remark 2.2.1. We have similar convention as the smooth manifold case for denoting sets of critical points. That is, the set of critical points of f and $\bar{f}$ are denoted $\mathrm{crit}(f)$ and $\mathrm{crit}(\bar{f})$

respectively, with $\mathrm{crit}_k(f)$ denoting critical points of index k. Note however that we say $\bar{p} \in \mathrm{crit}(\bar{f})$ if there exists $p \in \mathrm{crit}(f)$ such that $\pi(p) = \bar{p}$.

Definition 2.2.2. The *isotropy group* of a point $p \in X$ is defined as $G_p = \{g \in G : gp = p\}$.

Definition 2.2.3. If the G_p-action on the unstable manifold $W^u(p)$ at $p \in \pi^{-1}(\bar{p})$ is orientation preserving, then we say that p is an *orientable critical point*, and *non-orientable* otherwise. We write $\mathrm{crit}^+(\bar{f})$ (resp. $\mathrm{crit}^-(\bar{f})$) to denote the set of all orientable (resp. non-orientable) critical points of $\bar{f}$.

It is important to note that if G_p preserves the orientation of the unstable manifold of some $p \in \pi^{-1}(\bar{p})$, then $G_{p'}$ is also orientation preserving for all other critical points p' in the fiber.

2.2.2 Morse Complex for Global Quotients – Orientable Critical Points

As notation would suggest, let us denote $\mathrm{crit}_i^{\pm}(f) := \mathrm{crit}_i(f) \cap \mathrm{crit}^{\pm}(f)$. This allows us to decompose the vector spaces generated by the critical points of a Morse function as

$$CM_i(M, f) = CM_i^+(M, f) \oplus CM_i^-(M, f).$$

Surely, we have that a G-action preserves the orientable part of this decomposition, $CM_i^+(M, f)$, since we defined orientation preservation to only be determined by the isotropy group $G_p \leq G$.

Definition 2.2.4. For X a global quotient orbifold, we define $CM_i^+(X, \bar{f})$ to be the G-invariant part of $CM_i^+(M, f)$, which we denote $CM_i^+(M, f)^G$. For $\bar{p} \in \mathrm{crit}(\bar{f})$, we introduce the formal sum along the fiber $[\bar{p}] := \sum_{p \in \pi^{-1}(\bar{p})} p$, and we are able to say that these $[\bar{p}]$ form a basis for $CM^+(X, \bar{f}) = \bigoplus_i CM_i^+(X, \bar{f})$ for $\bar{p} \in \mathrm{crit}^+(\bar{f})$.

To form a chain complex, it remains to define the differential $\partial^i : CM_i^+(X, \bar{f}) \to CM_{i-1}^+(X, \bar{f})$. For each $\bar{p} \in \mathrm{crit}^+(\bar{f})$, let us fix G-invariant orientations on the unstable manifolds $W^u(p)$ for each $p \in \pi^{-1}(\bar{p})$. For a $\bar{q}$ which is not orientable, we can pick an arbitrary orientation on $W^u(q)$ for each $q \in \pi^{-1}(\bar{q})$.

Recall that for $p \in \mathrm{crit}_i(f)$, the Morse differential is $\partial p = \sum_{q \in \mathrm{crit}_{k-1}(f)} \#\mathcal{L}(p,q) \cdot q$. For an orbifold, we want to draw back on our formal sum of critical points along the fiber $\pi^{-1}(\bar{p})$ which was used to define the vector spaces generated by critical points of $\bar{f}$. So, for some $[\bar{p}]$, we set

$$\partial[\bar{p}] = \sum_{p \in \pi^{-1}(\bar{p})} \partial p.$$

This convention is natural: when we define $[\bar{p}]$, we aim to encode information about each point in the fiber above $\bar{p}$. Naturally then, the formal sum of the typical manifold differential applied to each point in the fiber defines the differential for $[\bar{p}]$ in a way which agrees with the definition of $CM^+(X, \bar{f})$.

In order to show that this defines a differential (i.e., $\partial^2 = 0$), we have the following two lemmata.

Lemma 2.2.1. If $\bar{p} \in \mathrm{crit}^+(\bar{f})$, then $\partial[\bar{p}] \in CM^+(X, \bar{f})$. In other words, $\partial[\bar{p}]$ has nonzero coefficients only at the orientable critical points of f.

Proof. Suppose $\mathrm{ind}(\bar{p}) = i$. We have then that each $p \in \pi^{-1}(\bar{p})$ is also of index i. We shall show that the coefficient in $\partial[\bar{p}]$ at an arbitrary $q \in \mathrm{crit}^-_{i-1}(f)$ is 0. As notation may suggest, we set $\mathcal{L}(\bar{p}, q) = \bigcup_{p \in \pi^{-1}(\bar{p})} \mathcal{L}(p, q)$.

Recall first that we fix an orientation of M, and of each unstable manifold. This automatically orients stable manifolds, so that for each $p, q \in \mathrm{crit}(f)$, one has that the intersection $W^-(p) \cap W^+(q)$ admits an induced orientation.

Take $\gamma \in \mathcal{L}(p, q)$ where $\mathrm{ind}(q) = \mathrm{ind}(p) - 1$. Then $\mathrm{im}\gamma \subset W^-(p) \cap W^+(q)$. If the orientation of γ matches the induced orientation of the intersection, then we count it as 1, and otherwise, as -1.

We can also express this sign rule as follows. If $s \in \mathbb{R}$ is a regular value between $f(q) < s < f(p)$m then we orient the set $f^{-1}(s)$ so that $[\nabla f][f^{-1}(s)] = [M]$, where the brackets denote the oriented frames of tangent spaces. Consider the set $S^- = W^-(p) \cap f^{-1}(s)$, which is oriented as a boundary of $D^-(p) = W^-(p) \cap f^{-1}([s, \infty))$. We follow the same convention for $S^+(q) = \partial D^+(q)$. It turns out that respectively, $S^-(p)$ and $S^+(q)$ are diffeomorphic to S^{i-1} and S^{n-i}. Since these sets are of complimentary dimension in $f^{-1}(s)$, we can count their signed intersection number. This matches the intersection number described in the previous paragraph.

For the proof of the lemma, suppose that $q \in \text{crit}^-$ such that $\text{ind}(q) = \text{ind}(p) - 1$. We can split $\mathcal{L}(\bar{p}, q) = \mathcal{L}(\bar{p}, q)^+ \sqcup \mathcal{L}(\bar{p}, q)^-$ with respect to their signs. It remains to show that $|\mathcal{L}(\bar{p}, q)^+| = |\mathcal{L}(\bar{p}, q)^-|$, which establishes that the number of "positives" equals the number of "negatives," that is, the coefficient is 0.

Pick some $g \in G_q$ which reverses the orientation of $W^-(q)$. Then g admits a permutation of $\mathcal{L}(\bar{p}, q)$, since g preserves the fiber above $\bar{p}$. We claim that the action of g maps $\mathcal{L}(\bar{p}, q)^+ \mapsto \mathcal{L}(\bar{p}, q)^-$. With the sign rule described, we have that $S^-(p)$ and $S^+(q)$ intersect positively at x in $f^{-1}(s)$, which means that the oriented frames at x agree, that is,

$$[S^-(p)]_x [S^+(q)]_x = [f^{-1}(s)]_x.$$

Since $\bar{p}$ is orientable, the choice of G-invariant orientations on the $W^-(p)$'s gives that $g \cdot [S^-(p)]_x = [S^-(p')]_{gx}$. Further, since g reverses the orientation of the unstable manifold at q, $g \cdot [S^+(q)]_x = -[S^+(q)]_{gx}$. As g preserves the orientation of F and since f is g-invariant, g preserves the orientation of $f^{-1}(s)$, or equivalently, $g \cdot [f^{-1}(s)]_x = [f^{-1}(s)]_{gx}$. So,

$$[S^-(p')]_{gx} [S^+(q)]_{gx} = (g \cdot [S^-(p)]_x)(-g \cdot [S^+(q)]_x) = -g \cdot [f^{-1}(s)]_x = -[f^{-1}(s)]_{gx}.$$

Thus at gx, we have that $S^-(p') \pitchfork S^+(q)$, and so the sign of the trajectory of $g \cdot \gamma$ is negative. The same argument holds for g^{-1}, and so $|\mathcal{L}(\bar{p}, q)^+| = |\mathcal{L}(\bar{p}, q)^-|$. □

Furthermore, we have that the differential is invariant under group action as long as $\bar{p}$ is orientable.

Lemma 2.2.2. $\partial[\bar{p}]$ is G-invariant if $\bar{p}$ is orientable.

Proof. The previous lemma gives that $CM^+(X, \bar{f})$ consists only of orientable critical points. Consider two orientable points q and $q' = g \cdot q$ in $CM^+(X, \bar{f})$, for $g \neq 1$ since we are assuming the group acts effectively. We want to show that the coefficients of q and q' are equal.

This is clear since g and g^{-1} provide the sign preserving isomorphisms between $\mathcal{L}(\bar{p}, q)$ and $\mathcal{L}(\bar{p}, q')$, since we chose the orientation on the unstable manifolds to be G-invariant. □

These lemmata establish the fact that ∂ preserves $CM_\bullet^+(M, f)^G$ as a subcomplex of $CM_\bullet(M, f)$, and we conclude that $CM_\bullet^+(X, \bar{f})$ is a subcomplex of $CM_*(M, f)$.

Definition 2.2.5. We write $CM_\bullet(\mathbf{X}, \bar{f})$ in place of $CM_\bullet^+(X, \bar{f})$ and use the same notation ∂ when we refer to the restriction of $\partial : CM_\bullet(M, f) \to CM_{\bullet-1}(M, f)$ to $CM_\bullet(\mathbf{X}, \bar{f})$. This differential inherits the property that $\partial^2 = 0$, and therefore forms a chain complex.

As one would enjoy, it does indeed turn out that the homology of the chain complex described is isomorphic to the singular homology of the quotient space X.

Theorem 2.2.1. $HM_\bullet(\mathbf{X}, \bar{f}) \cong H_\bullet(M/G) = H_\bullet(X)$.

Example 2.2.1. Consider the dented sphere again. We can equip S^2 with the Morse function $f : S^2 \to \mathbb{R}$, which is defined as the height function as seen in Example 1.1.1. We can equip this with a $\mathbb{Z}/2\mathbb{Z}$ action, which acts as a 180-degree rotation which identifies the two maxima, p and q, as seen in Figure 2.2. Notice that the quotient space is $[S^2/(\mathbb{Z}/2\mathbb{Z})]$ is topologically equivalent to S^2, and so we expect the Morse homology $HM_*([S^2/(\mathbb{Z}/2\mathbb{Z})])$ to be isomorphic to $H_*(S^2)$.

A chain complex which is obtained by the G-action on the critical points *without* considering orientability is

$$0 \longrightarrow \langle (p+q) \rangle \longrightarrow \langle r \rangle \longrightarrow \langle s \rangle \longrightarrow 0.$$

We denote by $\langle (p+q) \rangle$ the one dimensional vector space generated by the sum of the index-2 critical points p and q (it is only one dimensional since we identify these critical points under the group action). Notice, however, that the differential does not square to 0, and so homology is not defined.

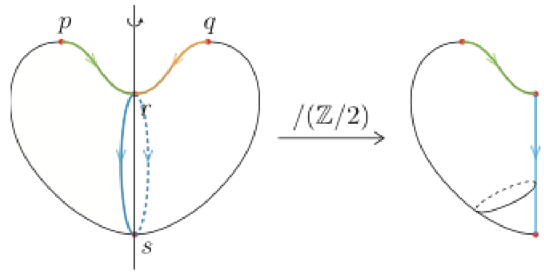

Figure 2.2: *The dented sphere with $\mathbb{Z}/2\mathbb{Z}$-action equipped. Image from [CH14].*

The correct complex must omit generators which are non-orientable critical points. Notice that since the half-rotation reverses the orientation of the unstable manifold at r, it is non-orientable. Thus, the proper chain complex should be

$$0 \longrightarrow \langle (p+q) \rangle \longrightarrow 0 \longrightarrow \langle s \rangle \longrightarrow 0.$$

The differential on this chain complex squares to 0, which defines homology. ◇

Remark 2.2.2. Note, the method of taking invariant subcomplexes does not work for general orbifolds which are not global quotients.

2.2.3 Morse Complex for Global Quotients – Orientation Spaces

There exists a second definition of the Morse complex for a global quotient orbifold, which in fact is a bit more natural. Such a definition relies on the "orientation space" which one assigns to each critical point.

Definition 2.2.6. We define the orientation space Θ_p^- at a critical point $p \in \mathrm{crit}(f)$ by

$$\Theta_p^- = \bigwedge^{\mathrm{ind}(p)} T_p W^-(p), \quad \text{or equivalently} \quad \Theta_p^- = H_{\mathrm{ind}(p)-1}(W^-(p) \setminus \{p\}),$$

a space which is isomorphic to $\mathbb{R}$.

Geometrically, we care about orientation spaces because a critical point p essentially represents the cell which comes from the unstable manifold $W^-(p)$, and therefore we should consider the group action $p \mapsto g \cdot p$ as a map $W^-(p) \mapsto W^-(g \cdot p)$. This map may or may not be orientation preserving, however. But, by introducing Θ_p^-, we can define a G-action on the Morse complex which includes the orientation data inherently. Indeed, we can redefine the Morse vector spaces as follows.

Definition 2.2.7. The Morse vector space can be expressed in terms of orientation spaces,

$$CM_*(M, f; \Theta) := \bigoplus_{p \in \text{crit}(f)} \Theta_p^-.$$

It remains to redefine the Morse boundary operator for the above expression of the Morse vector spaces. First, we fix an orientation for Θ_p^- for each $p \in \text{crit}(f)$. This gives a trivialization $\Theta_p^- \xrightarrow{\sim} \mathbb{R}\langle p \rangle$, sending the unit vector in the positive direction $1 \mapsto 1 = 1 \cdot p \in \mathbb{R}\langle p \rangle$. This gives an isomorphism

$$\varphi : CM_*(M, f; \Theta) \to CM_*(M, f),$$

which we can use to pull back the usual Morse boundary operator. That is,

$$\varphi^* \partial : CM_*(M, f; \Theta) \to CM_{*-1}(M, f)$$

defines a differential boundary map which agrees with this notion of orientation spaces generating the Morse vector spaces.

See now that $CM_*(M, f; \Theta)$ admits a G-action since the action $g : W^-(p) \to W^-(g \cdot p)$ induces an isomorphism $g_* : \Theta_p^- \to \Theta_{g \cdot p}^-$ defined as $g \cdot (p, p') = (g \cdot p, g_* p')$. If we restrict the metric on the ambient space, we have a G-invariant metric on each Θ_p^- to where $\|p'\| = \|g_* p\|$, meaning that $g \cdot (p, p') = (p, \pm p')$ for some $g \in G_p$, we see that we have $G_{(p,p')} = G_p$ if and only if p is orientable. Finally, we see that the restriction of the Morse vector spaces to the orbifold $\mathbf{X}$ agree with the previous definition from the previous section.

Lemma 2.2.3. The chain complex $CM_*(\mathbf{X}, f)$ is the same as the G-invariant subcomplex $CM_*(M, f; \Theta)^G$.

Proof. We have shown already that the two chain complexes agree on components generated by orientable critical points. Recall that $CM_*(\mathbf{X}, f) = CM_*^+(M, f)^G$. It can be seen from Lemma 2.2.1 that if $p \in \text{crit}(f)$ is non-orientable, then the component

$$\bigoplus_{g \cdot p \in \pi^{-1}(\bar{p})} \Theta^-_{g \cdot p} \in CM_*(M, f; \Theta)$$

is cancelled out after taking G-invariants. □

2.2.4 Intrinsic Form for the Differential

It turns out that compared to the formula of ∂ from before, we can construct a definition which is more "intrinsic" to the global quotient orbifold $\mathbf{X}$. This will become useful when we extend the results to general orbifolds in the next section. In particular, we will introduce the idea of "weights," in which the number of gradient flow lines between two orbifold critical points are counted with multiplicity depending on the number of lifts from $X \to M$ of a flow trajectory.

Consider $\bar{p}, \bar{q} \in \text{crit}(\bar{f})$ of indices k and $k+1$ respectively. Also suppose there exists a negative gradient flow line $\bar{\gamma}$ from $\bar{p}$ to $\bar{q}$ in X. We would like to see what the contribution of this flow $\bar{\gamma}$ is to the coefficient of $[\bar{q}]$ in the differential $\partial[\bar{p}]$. Take γ to be a lifting of $\bar{\gamma}$. Recall that a Morse function on an orbifold is defined to be a G-invariant function whose local lifts are Morse functions in the usual sense, and that the same idea holds for critical points, where $[\bar{p}] = \sum_{p \in \pi^{-1}(p)} p$.

Lemma 2.2.4. For any negative gradient flow line γ in M, the isotropy groups G_x for all points $x \in \text{im}\gamma$ are the same, and we label such group G_γ.

Proof. Note first that the diffeomorphism $\Phi_t : M \to M$ induced by the negative gradient vector field is G-equivariant. So, if $x \in \gamma$, suppose also $y \in \gamma$. Neither x nor y are critical points of f, and therefore there exists some $t \in \mathbb{R}$ such that $\Phi_t(x) = y$ (that is, we can flow along γ from x to y under some function of time). Equivariance gives that $\Phi_t(gx) = g\Phi_t(x) = gy$ for any $g \in G$ and so $G_x = G_y =: G_\gamma$. □

Since we have a notion of a common isotropy group along a flow "upstairs," we can define the conjugacy class represented by G_γ in G as $G_{\bar{\gamma}}$ "downstairs." Further, notice that the number of lifts of $\bar{\gamma}$ in M is given by the quantity $|G|/|G_{\bar{\gamma}}|$.

This means that there must exist $\sum_{\bar{\gamma}:\bar{p}\to\bar{q}} |G|/|G_{\bar{\gamma}}|$ negative flow lines connecting critical points which project down to $\bar{p}$ and $\bar{q}$. Now we want the coefficient of $[\bar{q}]$ instead of that of a single q, and so we divide the aforementioned sum by $|\pi^{-1}(\bar{q})|$. So, denoting by $G_{\bar{q}}$ the conjugacy class of G_q in X with q in the fiber of $\bar{q}$, and where $\varepsilon(\bar{\gamma}) = \pm 1$ depending on if the gradient flow orientation of $\bar{\gamma}$ matches the induced orientation $(+1)$ or not (-1),

$$\begin{aligned}
\partial[\bar{p}] &= \sum_{\bar{q}\in \mathrm{crit}^+_{i-1}(\bar{f})} \sum_{\bar{\gamma}:\bar{p}\to\bar{q}} \varepsilon(\bar{\gamma}) \frac{1}{|\pi^{-1}(\bar{q})|} \frac{|G|}{|G_{\bar{\gamma}}|}[\bar{q}] \\
&= \sum_{\bar{q}\in \mathrm{crit}^+_{i-1}(\bar{f})} \sum_{\bar{\gamma}:\bar{p}\to\bar{q}} \varepsilon(\bar{\gamma}) \frac{|G_{\bar{q}}|}{|G|} \frac{|G|}{|G_{\bar{\gamma}}|}[\bar{q}] \\
&= \sum_{\bar{q}\in \mathrm{crit}^+_{i-1}(\bar{f})} \sum_{\bar{\gamma}:\bar{p}\to\bar{q}} \varepsilon(\bar{\gamma}) \frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|}[\bar{q}]
\end{aligned}$$

Effectively, we must assign a weight based on the number of possible lifts of a path in X in our definition of $n(\bar{p},\bar{q})$ in order to define the differential boundary map. Define by $\nu_{\bar{q}}(\bar{\gamma}) := \varepsilon(\bar{\gamma})\frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|}$. On a minimal chart around $\bar{q}$, the preimage of $\bar{\gamma}$ is $|\nu_{\bar{q}}(\bar{\gamma})|$ copies of gradient flow lines which are obtained by the $G_{\bar{q}}$ action to a single lifting γ. Thus we think of $\nu_{\bar{q}}(\bar{\gamma})$ as the multiplicity or weight of $\bar{\gamma}$ at $\bar{q}$. Our definition of $\partial[\bar{p}]$ as above is indicative of the usual definition, $\partial p = \sum n(p,q)q$, therein we have that $n(\bar{p},\bar{q}) = \sum \nu_{\bar{q}}(\bar{\gamma})$. We can think of the number $n(\bar{p},\bar{q})$ then as the number of negative gradient flow lines between the two points counted with weight. Similarly, we think of $\nu_{\bar{p}}(\bar{\gamma}) = \varepsilon(\bar{\gamma})\frac{G_{\bar{p}}}{G_{\bar{\gamma}}}$ as the number of liftings of $\bar{\gamma}$ in a uniformizing chart around $\bar{p}$, counted with signs. Indeed, taking $\bar{p} = [\bar{p}]$ for ease of notation,

$$\partial\bar{p} = \sum_{\bar{q}} \left(\sum_{\bar{\gamma}:\bar{p}\to\bar{q}} \nu_{\bar{q}}(\bar{\gamma}) \right) \bar{q} = \sum_{\bar{q}} n(\bar{p},\bar{q})\bar{q}.$$

Note that the coefficients are intrinsic i.e. they only involve data of critical points of

$\bar{f}$, gradient flow lines in the quotient space, and local groups at critical points. Therefore we know that $\nu_{\bar{q}}(\bar{\gamma})$ and $n(\bar{p}, \bar{q})$ make sense for an arbitrary orbifold $\mathbf{X}$ and Morse-Smale function f. Notice of course that if the group action is trivial, meaning that our space remains a manifold, we get the standard definition of the Morse boundary operator. This will let us define a Morse complex for general orbifolds in the next section.

Similarly as before, we can define an alternative formula for differential boundary maps, where we take

$$\langle \bar{p} \rangle := \frac{|G_{\bar{p}}|}{|G|} \sum_{p \in \pi^{-1}(\bar{p})} p$$

instead of $[\bar{p}]$. We consider $\langle \bar{p} \rangle$ to be the average of p's with respect to the G action. Thus, the boundary operator is

$$\underline{\partial} \bar{p} := \partial \langle \bar{p} \rangle = \sum_{\bar{q} \in \mathrm{crit}^{+}_{i-1}(\bar{f})} \sum_{\bar{\gamma}: \bar{p} \to \bar{q}} \nu_{\bar{q}}(\bar{\gamma}) \cdot \langle \bar{q} \rangle.$$

The resulting homology group with $\underline{\partial}$ is isomorphic to the homology with ∂ via the map $\psi : \bar{p} \to |G_{\bar{p}}| \cdot \bar{p}$, as the only difference is that $\langle \bar{p} \rangle$ is thought of as an average, so we divide $|G|$ by the cardinality of the orbit of $\bar{p}$.

2.3 Morse-Smale-Witten Complex for General Orbifolds

Later in [CH14], Morse complexes are extended to general orbifolds, that is, not necessarily those which are global quotients. The main goal is to show that they are indeed chain complexes (i.e. $\partial^2 = 0$). When we work with smooth manifolds, this can be shown by studying the compactifications of negative gradient flow lines between critical points of index difference 2. However, although we can compactify connected components of the moduli spaces of trajectories on orbifolds, a broken trajectory (that is, a trajectory which makes "pit-stops" at different critical points down to its final destination) which represents ∂^2 on the orbifold can become a limit of several families of trajectories which are distinct, even after modding out by the group action. The solution to this issue is to add weights to the formula for the differential operator ∂. Indeed, doing so satisfies the condition that $\partial^2 = 0$. Finally, it is shown that the Morse homology with these weighted differential operators is again isomorphic to the singular homology of the underlying quotient space of the orbifold, mirroring Theorem 1.5.1.

2.3.1 Moduli Spaces of Gradient Flows

Throughout, consider $\mathbf{X}$ to be a compact, oriented, n-dimensional effective orbifold, which now may *not* necessarily be a global quotient. It is still possible to choose a Morse function in the sense of Definition 2.2.1. Also, we denote by X the underlying space of the quotient space $\mathbf{X}$. We now add the definition of the Smale condition for orbifolds.

Definition 2.3.1. A Morse function $\bar{f} : \mathbf{X} \to \mathbb{R}$ satisfies the Smale condition (is Morse-Smale) if for $\bar{p}, \bar{q} \in \text{crit}(\bar{f})$ and for $x \in W^-(\bar{p}) \cap W^+(\bar{q})$, we have the following transversality requirement:

$$T_x W^-(\bar{p}) \oplus T_x W^+(\bar{q}) = T_x \mathbf{X}.$$

Remark 2.3.1. Throughout, we assume that $\bar{f} : \mathbf{X} \to \mathbb{R}$ is a Morse-Smale function, which is induced by $f : X \to \mathbb{R}$.

For an orbifold with effective group actions, we can study the analytic properties of $\bar{f}$ through the frame bundle $Fr(\mathbf{X})$, which itself is a smooth manifold with a smooth, effective, almost free action of $O(n)$. Thus, we have that $\mathbf{X} \cong [Fr(\mathbf{X}), O(n)]$.

Definition 2.3.2. We say that a topological group action is locally free if each isotropy group (i.e. $G_x := \{g \in G : gx = x\}$) is discrete, that is, it is equipped with the discrete topology.

Consider a compact Lie group G which acts on a manifold M smoothly, effectively, and locally freely. Let $X = [M/G]$ and consider the orbifold Morse-Smale function $\bar{f} : X \to \mathbb{R}$.

We can lift $\bar{f}$ to $\tilde{f} : M \to \mathbb{R}$ via the factorization

$$\begin{array}{ccc} M & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\tilde{f}} & \\ \mathbf{X} & \xrightarrow[\bar{f}]{} & \mathbb{R} \end{array}$$

Lemma 2.3.1. $\tilde{f} : M \to \mathbb{R}$ as in the diagram above is a *Morse-Bott function*, that is it is a smooth function on a manifold where $\operatorname{crit}(\tilde{f})$ is a closed submanifold and whose Hessian is non-degenerate in the normal direction; equivalently, the kernel of the Hessian at a critical point equals the tangent space to the critical submanifold. Further, it satisfies the Smale transversality condition which is inherited from $\bar{f}$ through the lift.

Proof. We prove just the transversality claim. Consider two critical manifolds (or orbits) $O_{\bar{p}}$ and $O_{\bar{q}}$ associated to $\bar{p}, \bar{q} \in \operatorname{crit}(\bar{f})$. We write $W^-(O_{\bar{p}})$ and $W^+(O_{\bar{q}})$ to denote the unstable/stable manifolds of $O_{\bar{p}}$ and $O_{\bar{q}}$. For some $x \in W^-(O_{\bar{p}}) \cap W^+(O_{\bar{q}})$, we aim to show the transversality condition given in Definition 2.3.1. Let O_x be the G-orbit which contains x. Since both $W^-(O_{\bar{p}})$ and $W^+(O_{\bar{q}})$ are closed under the action of G, we have that $T_xO_x \subset T_xW^-(\bar{p}) \oplus T_xW^+(\bar{q})$. It remains to check in the normal direction to O_x. □

It turns out that Morse-Bott functions behave very nicely when studied in the context of Morse theory, that is, they are gluable and have nice convergence properties (cf. [AB95]). We would like to utilize these properties of $\tilde{f}$ in order to show that the connected components of the moduli space of gradient flow trajectories of $\bar{f}$ can be compactified.

Proposition 2.3.1. Consider $\bar{p}, \bar{q} \in \operatorname{crit}^+(\bar{f})$ with index difference 2. Then any connected component of $\mathcal{L}(\bar{p}, \bar{q})$ of $\bar{f} : X \to \mathbb{R}$ can be compactified by adding broken trajectories, which is necessarily a compact, 1-dimensional orbifold. Further, the uncompactified moduli space admits a natural orbifold structure (albeit noncompact of course).

Proof. Consider the negative gradient flow trajectories from $\bar{p}$ to $\bar{q}$, and denote by $\mathcal{L}(\bar{p}, \bar{q})$ the set of all such flow trajectories modded out by time (i.e., the usual notation). The strategy of the proof is to compactify a connected component of $\mathcal{L}(\bar{p}, \bar{q})$, which we denote by $\mathcal{P} \subset \mathcal{L}(\bar{p}, \bar{q})$, whose compactification we denote $\bar{\mathcal{P}}$. We will write it to be a quotient of a compact manifold with boundary by a locally free group action which preserves the boundary.

If we denote by $\tilde{\mathcal{P}} \subset \mathcal{L}(O_{\bar{p}}, O_{\bar{q}})$ the preimage of flow trajectories from $\bar{p}$ to $\bar{q}$ contained in the connected component $\mathcal{P}$ under the projection $\pi : M \to \mathbf{X}$, it can be expressed as a union of multiple connected components of $\mathcal{L}(O_{\bar{p}}, O_{\bar{q}})$ which is closed under G-action. This is because the G-action on M is locally free, and because $\mathcal{P}$ is a connected component of the global quotient $[\mathcal{L}^{\text{orb}}(O_{\bar{p}}, O_{\bar{q}})/G]$.

We can then describe $\mathcal{P}$ as a quotient of some component $\tilde{\mathcal{P}}_0$ of $\tilde{\mathcal{P}}$ under the action of some $H \leq G$. Computation of dimensions of unstable and stable manifolds then give us that $\dim(\mathcal{L}(O_{\bar{p}}, O_{\bar{q}})) = \dim G + 1$, meaning that $\dim \tilde{\mathcal{P}}_0 = \dim G + 1$.

The group action of G on $\mathcal{L}(O_{\bar{p}}, O_{\bar{q}})$ is orientation preserving since it preserved the orientation of the unstable/stable manifolds, and so the subgroup H preserves the orientation as well.

$\tilde{\mathcal{P}}_0$ is then compactified by adding broken trajectories from $\bar{p}$ to $\bar{q}$. Suppose the broken trajectory (γ_1, γ_2) is contained in the compactification $\tilde{\mathcal{P}}_0^{cpt}$ of $\tilde{\mathcal{P}}_0$. Then there exists a family of trajectories $\{\gamma_t\}_t \subset \tilde{\mathcal{P}}_0$ converging to (γ_1, γ_2). The H-invariance of $\tilde{f}$ and $\tilde{\mathcal{P}}_0$ imply that the H-action of $\tilde{\mathcal{P}}_0$ can be extended to its compactification $\tilde{\mathcal{P}}_0^{cpt}$.

In generality, $\tilde{\mathcal{P}}_0^{cpt}$ is a manifold with corners, however there are only codimension 1 strata in our case since the indices of $\bar{p}$ and $\bar{q}$ differ by 2. Note H preserves $\tilde{\mathcal{P}}_0^{cpt} \setminus \tilde{\mathcal{P}}_0$ since broken trajectories are sent to broken trajectories. So, since $\mathcal{P} \cong [\tilde{\mathcal{P}}_0/H] \subset [\tilde{\mathcal{P}}_0^{cpt}/H]$, we can consider $\bar{\mathcal{P}} := [\tilde{\mathcal{P}}_0^{cpt}/H]$ to be a compactification of $\mathcal{P}$.

Lastly, we claim that if $h \in H$ fixes $[\gamma] \in \tilde{\mathcal{P}}_0^{cpt}$ (i.e., $h \cdot \gamma$ is a time translation of γ), then h fixes all points of γ. Suppose for the sake of contradiction that distinct $x, y \in M$ are both on γ, and $h \cdot x = y$. Since γ is a flow trajectory, we know $f(x) \neq f(y)$ by definition, contradicting the H-invariance of f. Therefore each $\gamma \in \bar{\mathcal{P}}$ has a finite isotropy group, and so $\bar{\mathcal{P}}$ is an orbifold with boundary whose interior is isomorphic to $\mathcal{P}$. It is 1-dimensional since $\dim G = \dim H$ (that is, $[G : H] < \infty$ since $\tilde{\mathcal{P}}_0$ is a maximal connected subset of $\tilde{\mathcal{P}}$), and $\dim \tilde{\mathcal{P}}_0^{cpt} = \dim G + 1$. □

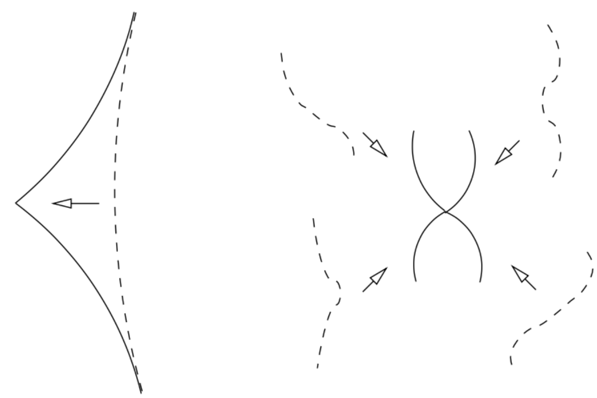

Figure 2.3: *Limits of gradient flows for manifolds (left) versus orbifolds (right). Image from [CH14].*

We can apply the same proof technique to achieve a result about flow trajectories between critical points of index difference 1, which is indeed just a special case of the above because we do not worry about broken trajectories.

Corollary 2.3.1. Consider $\bar{p}, \bar{q} \in \text{crit}^+(\bar{f})$ with index difference 1. Then there are only finitely many negative gradient flow trajectories from $\bar{p}$ to $\bar{q}$.

Note, from here on out, we write γ instead of $[\gamma]$ for simplification.

2.3.2 Morse-Smale-Witten Complex

Through the "intrinsic" form of the differential in which we consider weighted trajectories, we can construct a complex of orbifolds which are not global quotients. Suppose $\bar{f} : \mathbf{X} \to \mathbb{R}$ is Morse-Smale, and as usual, take the $\mathbb{R}$-vector spaces $CM_k(\mathbf{X}, \bar{f})$ to be generated by $\text{crit}^+k(\bar{f})$. Recall the notation of the intrinsic differential applied to some $\bar{p} \in \text{crit}^+_k(\bar{f})$,

$$\partial \bar{p} = \sum_{\bar{q} \in \text{crit}^+_{k-1}(\bar{f})} n(\bar{p}, \bar{q})\bar{q} = \sum_{\bar{q} \in \text{crit}^+_{k-1}(\bar{f})} \sum_{\bar{\gamma} \in \mathcal{L}(\bar{p}, \bar{q})} \nu_{\bar{q}}(\bar{\gamma})\bar{q}.$$

This section is entirely dedicated to the proof of the following theorem.

Theorem 2.3.1. $(CM_*(X, \bar{f}), \partial)$ defines a chain complex, i.e. $\partial^2 = 0$.

This is a nontrivial result. The difference between this and the simple smooth manifold case as in Theorem 1.3.1 will be explored. The standard argument used to show that $\partial^2 = 0$ on manifolds uses the compactifications of negative gradient flow lines between critical points of index difference 2 (cf. page 54 in [AD13]). The issue, however, is that although we can compactify connected components of moduli spaces of trajectories on orbifolds between critical points of index difference 2 (cf. Proposition 2.3.1), a broken trajectory which represents ∂^2 on X can become a limit of several families of trajectories which are distinct even after modding out by G. This is illustrated in Figure 2.3, and in the following example.

Example 2.3.1. Consider the broken trajectory $(\bar{\gamma}, \bar{\delta})$, where $\bar{\gamma}$ is a negative gradient flow line from $\bar{p}$ to $\bar{q}$, and $\bar{\delta}$ is from $\bar{q}$ to $\bar{r}$. Assume for the sake of simplicity that $G_{\bar{\gamma}} = G_{\bar{\delta}} = 1$. On some uniformizing neighborhood $(\tilde{U}_{\bar{q}}, G_{\bar{q}}, \pi_{\bar{q}})$, there are $|G_{\bar{q}}|$ flow lines which lift $\bar{\gamma}$ and also $|G_{\bar{q}}|$ flow lines which lift $\bar{\delta}$. Suppose γ and δ are each lifts which cover $\bar{\gamma}$ and $\bar{\delta}$ respectively in the cover $\tilde{U}_{\bar{q}}$. Then (γ, δ) lifts the broken trajectory $(\bar{\gamma}, \bar{\delta})$ in X. For each pair (g_1, g_2) in $G_{\bar{q}}$, we have that $g_1 \cdot \gamma$ along with $g_2 \cdot \delta$ give another broken trajectory in the cover which projects to $(\bar{\gamma}, \bar{\delta})$.

Thus, we find that there are $|G_{\bar{q}}|^2$ broken trajectories in $\tilde{U}_{\bar{q}}$ lying over $(\bar{\gamma}, \bar{\delta})$ and so there are $|G_{\bar{q}}|^2$ families of smooth gradient flow lines which converge to one of the $|G_{\bar{q}}|^2$ broken trajectories in $\tilde{U}_{\bar{q}}$. The assumption that $G_{\bar{\gamma}} = G_{\bar{\delta}} = 1$, the $G_{\bar{q}}$ action on the set of broken trajectories in $\tilde{U}_{\bar{q}}$ is free and therefore is on the set of local gluings in $\tilde{U}_{\bar{q}}$. So, there are $|G_{\bar{q}}| = |G_{\bar{q}}|^2 / |G_{\bar{q}}|$ families after quotienting by the $G_{\bar{q}}$ action. So, there are $|G_{\bar{q}}|$ distinct families of smooth trajectories which converge to a single broken trajectory $(\bar{\gamma}, \bar{\delta})$ near $\bar{q}$.

Note that this means that locally around a breaking point of the broken trajectory, we have that although each limiting trajectory (the lines leading to the breaking point) share the same limit, which is the broken trajectory, the orbifold structures of each limiting trajectory can be rather different. In Figure 2.4, we see that two limiting trajectories $\{\gamma_t^1\}$ and $\{\gamma_t^2\}$ have the same broken trajectory as a limit, however the orbifold structures along the flows can be different, for example, one may have influence from another group action elsewhere. This difference between structure at a limit point versus structure along the length of the

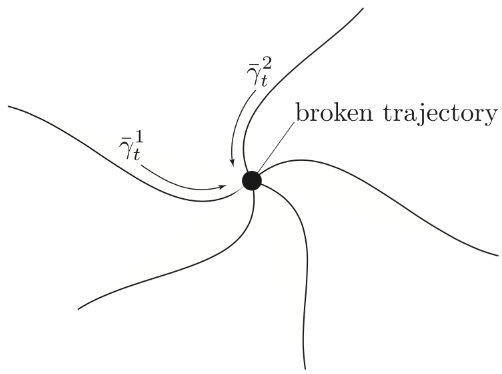

Figure 2.4: *Two limiting trajectories in the 1-dimensional moduli space of flow lines. Image from [CH14].*

various flows motivates the reason why we want to add "weights" to the formula for the orbifold Morse boundary operator.

Notation. Consider critical points $\bar{p} \in \text{crit}_k^+(\bar{f})$, $\bar{q}, \bar{q}' \in \text{crit}_{k-1}(\bar{f})$, $\bar{r} \in \text{crit}_{k-2}^+(\bar{f})$. Note $\bar{q}, \bar{q}'$ are not assumed to be orientable. Take $\bar{\gamma}$ (resp. $\bar{\gamma}'$) to be negative gradient flow lines from $\bar{p}$ to $\bar{q}$ (resp. $\bar{q}'$), and take $\bar{\delta}$ (resp. $\bar{\delta}'$) to be flow lines from $\bar{q}$ (resp. $\bar{q}'$) to $\bar{r}$. Suppose that two broken trajectories $(\bar{\gamma}, \bar{\delta})$ and $(\bar{\gamma}', \bar{\delta}')$ are connected by a 1-parameter family of negative gradient flow lines from $\bar{p}$ to $\bar{r}$. Denote the set of flow lines in the above 1-parameter family by $\mathcal{P}$.

Note that even if $\mathcal{P}$ flows between two orientable critical points, the breaking point (either $\bar{q}$ or $\bar{q}'$) of a broken trajectory in the limit is not necessarily orientable. This is why we do not care about orientability of these two points in our assumptions.

In order to prove Theorem 2.3.1, we must first develop some lemmata.

Lemma 2.3.2. $\mathcal{P}$ is a one dimensional oriented orbifold whose stabilizers $G_{\bar{\gamma}}$ are all isomorphic for each $\bar{\gamma} \in \mathcal{P}$. Thus, it is an ineffective orbifold (i.e. there exist elements in G other than 1 which act trivially) whenever $G_{\bar{\gamma}} \neq \{e\}$.

Proof. It follows from Proposition 2.3.1 that $\mathcal{P}$ is an oriented 1-dimensional orbifold. It suffices to show that all the stabilizers are isomorphic. This is almost trivial however; consider such an orbifold as an interval $(-1, 1)$. A finite group action on this interval is necessarily id or $x \mapsto -x$ up to diffeomorphism. The latter is certainly not orientation preserving, so we can not consider it; therefore local groups act trivially, and hence the stabilizers are isomorphic to each other. □

We know that there exists a compactification $\bar{\mathcal{P}}$ of each component $\mathcal{P} \subset \mathcal{L}^{\text{orb}}(\bar{p}, \bar{q})$, obtained by adding limit broken trajectories $(\bar{\gamma}, \bar{\delta})$ and $(\bar{\gamma}', \bar{\delta}')$ to $\mathcal{P}$ (cf. Proposition 2.3.1). We now consider the orbifold structure of $\bar{\mathcal{P}}$, in particular the stabilizers of the limiting trajectories and that of $\mathcal{P}$.

Consider the chart $(\tilde{U}_{\bar{q}}, G_{\bar{q}}, \pi_{\bar{q}})$ around $\bar{q}$ with $U_{\bar{q}} = \pi_{\bar{q}}(\tilde{U}_{\bar{q}})$. Take Γ to be the set of all liftings of $\bar{\gamma} \cap U_{\bar{q}}$ and Δ to be the set of all liftings of $\bar{\delta} \cap U_{\bar{q}}$. In particular, $\Gamma = \{\gamma \subset \tilde{U}_{\bar{q}} : \pi_{\bar{q}}(\gamma) = \bar{\gamma} \cap U_{\bar{q}}\}$, and $\Delta = \{\delta \subset \tilde{U}_{\bar{q}} : \pi_{\bar{q}}(\delta) = \bar{\delta} \cap U_{\bar{q}}\}$. Certainly since $G_{\bar{q}} \curvearrowright \tilde{U}_{\bar{q}}$, we have that $G_{\bar{q}} \curvearrowright \Gamma \times \Delta$ by the diagonal action (i.e. $(x, y) \mapsto (gx, gy)$). So, we can consider $(\Gamma \times \Delta)/G_{\bar{q}}$ to be the set of all possible smooth trajectories converging to $(\bar{\gamma}, \bar{\delta})$ in X.

Lemma 2.3.3. $\mathcal{P}$ determines an element of $(\Gamma \times \Delta)/G_{\bar{q}}$, say $[\gamma, \delta] \in (\Gamma \times \Delta)/G_{\bar{q}}$, and this correspondence is one-to-one locally around $\bar{q}$.

Consider the isotropy groups G_γ, G_δ of γ, δ. The intersection $G_\gamma \cap G_\delta \subset G_{\bar{q}}$ is regarded as the isotropy group at the boundary point $(\bar{\gamma}, \bar{\delta})$ of $\mathcal{P}$. We denote its conjugacy class by $G_{[\gamma,\delta]}$.

Proposition 2.3.2. The limit of isotropy groups are always a subgroup of the isotropy group at the limit point.

Corollary 2.3.2. The converse to Proposition 2.3.2 is true when we are considering the moduli space of gradient flow trajectories.

This corollary in particular will be crucial in the proof of Theorem 2.3.1. Note that the G_γ's are conjugate to each other for liftings γ of $\bar{\gamma}$, however the intersection $G_\gamma \cap G_\delta$ depends on the choice of lifts for both $\bar{\gamma}$ and $\bar{\delta}$. Further, its cardinality depends on these choices. Given that $\mathcal{P}$ determines an element of $(\Gamma \times \Delta)/G_{\bar{q}}$, we have a result similar to Lemma 2.2.4.

Lemma 2.3.4. $G_{[\gamma,\delta]} \cong G_{\bar{c}}$ for any $\bar{c} \in \mathcal{P}$.

This gives that $\bar{\mathcal{P}}$ is an ineffective orbifold. Further, it carries a natural orientation. We will show that the orientation at the boundary broken trajectories of $\bar{\mathcal{P}}$ oppose each other. We introduce the following lemma-definition in order to give a sign rule for the boundary of $\bar{\mathcal{P}}$ by showing that signs from $\Gamma \times \Delta$ are inherited by its quotient by $G_{\bar{q}}$.

Lemma 2.3.5. For $(\gamma, \delta) \in \Gamma \times \Delta$, let us define $\varepsilon(\gamma, \delta) := \varepsilon(\gamma) \cdot \varepsilon(\delta)$. Regardless of $\bar{q}$ being orientable or not, the $G_{\bar{q}}$ action on $\Gamma \times \Delta$ preserves $\varepsilon(\gamma, \delta)$. That is, $\varepsilon(g\gamma, g\delta) = \varepsilon(\gamma, \delta)$ for all $g \in G_{\bar{q}}$.

Proof. Suppose $\bar{q}$ is orientable, so that $\bar{\gamma}$ and $\bar{\delta}$ can be given well-defined signs. Then $\varepsilon[\gamma, \delta] = \varepsilon(\bar{\gamma}) \cdot \varepsilon(\bar{\delta})$ for all broken trajectories $[\gamma, \delta] \in (\Gamma \times \Delta)/G_{\bar{q}}$ since the $G_{\bar{q}}$-action preserves signs.

Suppose now that $g \in G_{\bar{q}}$ reverses the orientation of $W^{-}(\bar{q})$. Since $\bar{p}$ and $\bar{q}$ are both orientable, we can use the argument from the proof of Lemma 2.2.1 to get that $\varepsilon(g \cdot \gamma) = -\varepsilon(\gamma)$ and $\varepsilon(g \cdot \delta) = -\varepsilon(\delta)$. □

Therefore, we can define $\varepsilon[\gamma, \delta] := \varepsilon(\gamma, \delta) = \varepsilon(\gamma) \cdot \varepsilon(\delta)$. This allows us (by way of Lemma 2.3.3) to say that:

Lemma 2.3.6. If there is a 1-parameter family $\mathcal{P}$ which corresponds to $[\gamma, \delta]$ and $[\gamma', \delta']$, then $\varepsilon[\gamma, \delta]$ and $\varepsilon[\gamma', \delta']$ should be opposite.

This gives that $\partial^2 = 0$ in the smooth case, as the signs cancel. But in order to count gradient flow trajectories and to describe the cancellation in the orbifold case, we need to take weighted sums for the Morse boundary operator.

Definition 2.3.3. For the compactification $\bar{\mathcal{P}}$ as above, the following expression is the *weighted boundary of* $\bar{\mathcal{P}}$:

$$\partial \bar{\mathcal{P}} = \frac{\varepsilon[\gamma, \delta]}{|G_{[\gamma,\delta]}|}(\bar{\gamma}, \bar{\delta}) + \frac{\varepsilon[\gamma', \delta']}{|G_{[\gamma',\delta']}|}(\bar{\gamma}', \bar{\delta}') := \omega_{\mathcal{P}}(\bar{\gamma}, \bar{\delta}) + \omega_{\mathcal{P}}(\bar{\gamma}', \bar{\delta}').$$

The cofficients to $(\bar{\gamma}, \bar{\delta})$ and $(\bar{\gamma}', \bar{\delta}')$ are called weights.

The standard argument showing that $\partial^2 = 0$ in the smooth case together with the above weights give the following equation:

$$\sum_{(\bar{\zeta},\bar{\eta})\in\partial\mathcal{P}} \omega_{\mathcal{P}}(\bar{\zeta},\bar{\eta}) = 0.$$

Denote by $\overline{\mathcal{L}}(\bar{p},\bar{r})$ the compactified moduli space of negative gradient flow lines from $\bar{p}$ to $\bar{r}$. Geometrically this is given by several copies of compact intervals equipped with trivial actions of the corresponding isotropy groups which are possibly joined at boundary points if they define families of flow lines sharing the same limit. Note that the limiting flows to a fixed broken trajectory might have non-isomorphic stabilizers. Thus, this space is not an orbifold with boundary, which differs from the smooth case where its analogue is a manifold with corners.

Notation. Denote $\partial\overline{\mathcal{L}}(\bar{p},\bar{r}) = \overline{\mathcal{L}}(\bar{p},\bar{r}) \setminus \mathcal{L}(\bar{p},\bar{r})$.

Definition 2.3.4. If $(\bar{\zeta},\bar{\eta}) \in \partial\overline{\mathcal{L}}(\bar{p},\bar{r})$, we define

$$\omega(\bar{\zeta},\bar{\eta}) := \sum_{\mathcal{P}\text{ s.t. }(\bar{\zeta},\bar{\eta})\in\partial\bar{\mathcal{P}}} \omega_{\mathcal{P}}(\bar{\zeta},\bar{\eta}),$$

where the sum is taken over all all 1-parameter families $\mathcal{P}$ whose boundary contains $(\bar{\zeta},\bar{\eta})$. Finally, we denote the sum of all weights associated to the gluings converging to one of the broken trajectories through $\bar{p},\bar{q},\bar{r}$ as

$$\omega(\bar{p},\bar{q},\bar{r}) := \sum_{(\bar{\zeta},\bar{\eta})\in\mathcal{L}(\bar{p},\bar{q})\times\mathcal{L}(\bar{q},\bar{r})} \omega(\bar{\zeta},\bar{\eta}).$$

Note that $\mathcal{L}(\bar{p},\bar{q})\times\mathcal{L}(\bar{q},\bar{r}) \subset \partial\mathcal{L}(\bar{p},\bar{r})$ for any $\bar{q}$. Since all of the intervals contained in $\overline{\mathcal{L}}(\bar{p},\bar{r})$ are oriented so that they are compatible with their boundary orientations, all terms in the first sum in the definition have the same signs.

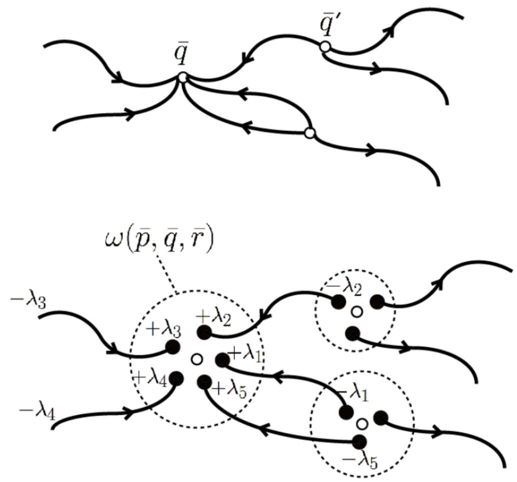

Figure 2.5: *1-dimensional moduli spaces near orientable critical points (top) and under the consideration of orbifold structures (bottom). Image from [CH14].*

The equation $\sum_{(\bar{\zeta},\bar{\eta})\in\partial\mathcal{P}} \omega_{\mathcal{P}}(\bar{\zeta},\bar{\eta}) = 0$ gives that

$$\sum_{\bar{q}\in \mathrm{crit}_{k-1}(\bar{f})} \omega(\bar{p},\bar{q},\bar{r}) = \sum_{(\bar{\zeta},\bar{\eta})\in\partial\overline{\mathcal{L}}(\bar{p},\bar{r})} \omega(\bar{\zeta},\bar{\eta}) = \sum_{\mathcal{P}} \sum_{(\bar{\zeta},\bar{\eta})\in\partial\bar{\mathcal{P}}} \omega_{\mathcal{P}}(\bar{\zeta},\bar{\eta}) = 0.$$

Notice in Figure 2.5 that paths represent oriented 1-dimensional moduli spaces which converge to broken trajectories. This shows how we are summing the weighted contributions near orientable critical points. In the bottom image, see that $\sum_{i=1}^{5} \lambda_i$ contributes to $\omega(\bar{p},\bar{q},\bar{r})$. Similarly, summing over one of the other dotted circles contributes to $\omega(\bar{p},\bar{q}',\bar{r})$. Near a non-orientable critical point, we have that a similar situation to Lemma 2.2.1 (i.e., non-orientable critical points do not contribute to coefficients in the differential), as seen specifically in Lemma 2.3.8.

Lemma 2.3.7. Let S be a finite set on which a finite group G acts. Suppose that S/G is a weighted set such that each element $[x]\in S/G$ has the weight $\lambda_{[x]}$. Then,

$$\sum_{[x]\in S/G} \lambda_{[x]} = \frac{1}{|G|}\sum_{x\in S} \lambda_{[x]}\cdot |G_x|.$$

Proof. The proof follows from the proof of Burnside's lemma. □

Finally, we give the proof to Theorem 2.3.1. We utilize the definition of the differential which takes weights in to account, and the computation is fairly straightforward, as can be seen.

Proof. Observe that by definition,

$$\nu_{\bar{r}}(\bar{\delta}) = \varepsilon(\bar{\delta}) \frac{|G_{\bar{r}}|}{|G_{\bar{\delta}}|} = \frac{|G_{\bar{r}}|}{|G_{\bar{q}}|} \cdot \nu_{\bar{q}}(\bar{\delta}).$$

Therefore, we have the following sequence of equalities (recall that we write $\bar{p} = [\bar{p}]$ for ease):

$$\begin{aligned}
\partial^2 \bar{p} &= \partial \left(\sum_{\bar{q} \in \mathrm{crit}^+_{k-1}(\bar{f})} \sum_{\bar{\gamma} \in \mathcal{L}(\bar{p}, \bar{q})} \nu_{\bar{q}}(\bar{\gamma}) \right) \\
&= \sum_{\bar{r} \in \mathrm{crit}^+_{k-2}(\bar{f})} \left(\sum_{\bar{q} \in \mathrm{crit}^+_{k-1}(\bar{f})} \sum_{(\bar{\gamma}, \bar{\delta}) \in \partial \mathcal{L}(\bar{p}, \bar{r})} \nu_{\bar{q}}(\bar{\gamma}) \nu_{\bar{r}}(\bar{\delta}) \right) \bar{r} \\
&= \sum_{\bar{r} \in \mathrm{crit}^+_{k-2}(\bar{f})} |G_{\bar{r}}| \left(\sum_{\bar{q} \in \mathrm{crit}^+_{k-1}(\bar{f})} \sum_{(\bar{\gamma}, \bar{\delta}) \in \partial \mathcal{L}(\bar{p}, \bar{r})} \frac{\nu_{\bar{q}}(\bar{\gamma}) \nu_{\bar{q}}(\bar{\delta})}{|G_{\bar{q}}|} \right) \bar{r} \\
&= \sum_{\bar{r} \in \mathrm{crit}^+_{k-2}(\bar{f})} |G_{\bar{r}}| \left(\sum_{\bar{q} \in \mathrm{crit}^+_{k-1}(\bar{f})} \omega(\bar{p}, \bar{q}, \bar{r}) \right) \bar{r} \\
&= \sum_{\bar{r} \in \mathrm{crit}^+_{k-2}(\bar{f})} |G_{\bar{r}}| \cdot 0 \cdot \bar{r} \\
&= 0.
\end{aligned}$$

$\square$

Lemma 2.3.8. If $\bar{q}$ is non-orientable, then $\omega(\bar{p}, \bar{q}, \bar{r}) = 0$, and if $\bar{q}$ is orientable, then

$$\sum_{(\bar{\gamma}, \bar{\delta})} \frac{\nu_{\bar{q}}(\bar{\gamma}) \nu_{\bar{q}}(\bar{\delta})}{|G_{\bar{q}}|} = \omega(\bar{p}, \bar{q}, \bar{r}).$$

Therefore, we have that

$$\sum_{\bar{q}\in \mathrm{crit}_{k-1}(\bar{f})} \omega(\bar{p},\bar{q},\bar{r}) = \sum_{\bar{q}\in \mathrm{crit}^+_{k-1}(\bar{f})} \omega(\bar{p},\bar{q},\bar{r}) = 0.$$

Remark 2.3.2. Note that the result that $\underline{\partial}^2 = 0$ is almost identical. Indeed, this is automatic since there is a $(\partial, \underline{\partial})$ chain map $\psi : \bar{p} \to |G_{\bar{p}}| \cdot \bar{p}$ which is an $\mathbb{R}$-linear isomorphism.

2.3.3 Morse Homology is Singular Homology

The goal of this section is to show that the homology of the Morse complex $(CM_*(X, \bar{f}), \partial)$ of general orbifolds equals the singular homology of the quotient space. Throughout, assume that the critical points of $\bar{f}$ are self-indexing, that is that $\bar{f}(\bar{p}_i) = \lambda_i$, the Morse index of $\bar{p}_i$.

Similar to Theorem 1.1.2, we can establish a similar theorem for handle decomposition for orbifolds. We will utilize the cell structure in order to show that the Morse-Smale-Witten complex of general orbifolds is indeed the singular homology of the product space.

Theorem 2.3.2 (Fundamental Theorem 2 for Orbifolds)**.** Let $\bar{p} \in \mathrm{crit}_k(\bar{f})$ and let $\bar{f}(\bar{p}) = c$. Suppose $\bar{p}$ is the unique critical point in the level set $\bar{f}^{-1}[c-\varepsilon, c+\varepsilon]$ for small $\varepsilon \ll 1/2$. Then we have the expected homotopy

$$\bar{f}^{-1}(-\infty, c+\varepsilon] \simeq \bar{f}^{-1}(-\infty, c-\varepsilon] \cup_{\partial D^k/G_{\bar{p}}} D^k/G_{\bar{p}}.$$

Here D^k is a small invariant disk in the unstable manifold at $\bar{p}$ in a uniformizing chart around $\bar{p}$, and hence endowed with the $G_{\bar{p}}$ action.

In order to compute homological information about attaching cells, we must note the following theorem regarding relative homology in the setting of equivariant topology; that is, the topology of spaces which possess symmetries imposed by groups.

Theorem 2.3.3 (Theorem 2.4 in [Bre72])**.** Let K be a regular G-simplicial complex (with G finite) and let L be a subcomplex of K. Then $H_*(K, L; \mathbb{R})^G \cong H_*(K/G, L/G; \mathbb{R})$, where the left hand side is the subset of $H_*(K, L; \mathbb{R})$ which is fixed by G.

A simple application of this is if we consider the n-dimensional disk D^n and the finite group Γ acting on $(D^n, \partial D^n)$. Then we have

$$H_i(D^n/\Gamma, \partial D^n/\Gamma; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = n \text{ and } \Gamma \text{ preserves orientation of } D^n, \\ 0 & \text{otherwise.} \end{cases}$$

In the smooth case, the coefficient of q in ∂p is defined by the relative intersection number between $W^-(p)$ and $W^+(q)$. It will be better to consider instead the integration of Thom forms. Per [CR04], the Thom form of a suborbifold $\mathbf{N} \subset \mathbf{X}$ is defined locally as the invariant Thom form of the preimage $\tilde{N}$ of $\mathbf{N}$ in each uniformizing chart. Take N to denote the underlying quotient space of $\mathbf{N}$. Also, on a uniformizing chart, the integral of a Thom form along a normal fiber of $\tilde{N}$ at $p \in \tilde{N}$ is 1 where $\pi(p) = \bar{p}$.

We will utilize a version of the Stokes theorem for orbifolds as well. A C^∞ singular simplex $\bar{s}$ of dimension k in $\mathbf{X}$ is defined by a smooth map $\bar{s}$ from the k-dimensional simplex Δ_k to X. Suppose that the image of $\bar{s}$ lies in a single uniformizing chart $(\tilde{U}, G, \pi)$ and it admits a lifting $s : \Delta_k \to \tilde{U}$ with $\pi \circ s = \bar{s}$. Consider a k-form ω on $\pi(\tilde{U})$, which is given by an invariant k-form ω on $\tilde{U}$. We define

$$\int_{\bar{s}} \bar{\omega} = \int_{\Delta_k} s^* \omega.$$

It is important to note however, that s is a singular chain in Euclidean space, and so weights are not defined on the right hand side. For a general $\bar{s}$, we use a partition of unity to define $\int_{\bar{s}} \bar{\omega}$. As desired, we have that the Stokes theorem still holds for $\bar{\omega}$.

Definition 2.3.5. $\underline{\mathrm{W}}^{\pm}(\bar{p}) = \{\bar{x} \in X : \lim_{t \to \pm\infty} \bar{\Phi}_t(\bar{x}) = \bar{p}\}$ denotes the unstable and stable manifolds on orbifolds.

Theorem 2.3.4. The homology constructed in Subsection 2.3.2 is isomorphic to the singular homology of the underlying space X of $\mathbf{X}$.

Proof. Consider first the filtration of singular homology defined in a way which is inspired by the cellular decomposition of Theorem 2.3.2. For a small ε, take

$$X_k = \bar{f}^{-1}((-\infty, k+1-\varepsilon)) \quad \text{and} \quad Y_k = \bar{f}^{-1}([k-\varepsilon, k+1-\varepsilon]).$$

This induces the following filtration of the singular chain complex:

$$C_*(X_0;\mathbb{R}) \subset C_*(X_1;\mathbb{R}) \subset \cdots \subset C_*(X_n;\mathbb{R}) = C_*(X;\mathbb{R})$$

Let us set $\underline{\mathrm{D}}^{\pm}(\bar{p}) = \underline{\mathrm{W}}^{\pm}(\bar{p}) \cap Y_k$. Since ε is close to 0, we have that topologically, $\underline{\mathrm{D}}^{\pm}(\bar{p}) \cong D^{\pm}(p)/G_{\bar{p}}$, where the $D^{\pm}(p)$ are small invariant neighborhoods of $p \in \pi^{-1}(\bar{p})$ contained in the stable/unstable manifolds of p with respect to the lifting f of $\bar{f}$. Further, we write $\partial\underline{\mathrm{D}}^{\pm}(\bar{p})$ to be the image of $\partial D^{\pm}(p)/G_{\bar{p}}$.

Via the excision property of singular homology, we can write

$$H_*(X_k, X_{k-1};\mathbb{R}) = \begin{cases} \bigoplus_{\bar{p}\in \mathrm{crit}_k(\bar{f})} H_k(\underline{\mathrm{D}}^-(\bar{p}), \partial\underline{\mathrm{D}}^-(\bar{p});\mathbb{R}) & * = k, \\ 0 & \text{otherwise.} \end{cases}$$

From the given application of Theorem 2.3.3, we see that

$$H_k(\underline{\mathrm{D}}^-(\bar{p}), \partial\underline{\mathrm{D}}(\bar{p});\mathbb{R}) \cong H_k(D^-(p), \partial D^-(p))^{G_{\bar{p}}},$$

where if $\bar{p}$ is orientable, it is isomorphic to $\mathbb{R}$, and 0 otherwise.

We can utilize the E^1 terms of the spectral sequence derived from our filtered chain complex in order to produce the chain complex

$$\cdots \longrightarrow H_{k+1}(X_{k+1}, X_k;\mathbb{R}) \longrightarrow H_k(X_k, X_{k-1};\mathbb{R}) \longrightarrow H_{k-1}(X_{k-1}, X_{k-2};\mathbb{R}) \longrightarrow \cdots,$$

whose boundary map is defined via the composition

$$H_k(X_k, X_{k-1}; \mathbb{R}) \longrightarrow H_{k-1}(X_{k-1}; \mathbb{R}) \longrightarrow H_{k-1}(X_{k-1}, X_{k-2}; \mathbb{R}).$$

Now, we would like to choose a generator of $H_k(X_k, X_{k-1}; \mathbb{R})$ which is roughly a smooth singular chain in X_k and represents $(\underline{\mathrm{D}}^-(\bar{p}), \partial\underline{\mathrm{D}}^-(\bar{p}))$, where $\bar{p} \in \mathrm{crit}_k^+(\bar{f})$. Taking again a sufficiently small ε, we may assume that $\underline{\mathrm{D}}^-(\bar{p}) \subset \pi_{\bar{p}}(\tilde{U}_{\bar{p}})$. Then we have a neighborhood $D^-(p)$ where $p = \pi_{\bar{p}}^{-1}(p)$ in the unstable manifold of the lifting of f, which covers $\underline{\mathrm{D}}^-(\bar{p})$. This $D^-(p)$ represents a smooth singular chain in $\tilde{U}_{\bar{p}}$, which is a map $|p\rangle$ from the formal sum of simplices in Euclidean space to the uniformizing chart $\tilde{U}_{\bar{p}}$.

The composition $\pi_{\bar{p}} \circ |p\rangle$ "wraps" $\underline{\mathrm{D}}^-(\bar{p})$ $|G_{\bar{p}}|$ times. We can adjust the map by the reciprocal of the weight in order to define $|\bar{p}\rangle = |G_{\bar{p}}|^{-1}\pi_{\bar{p}} \circ |p\rangle$, and we will use this $|\bar{p}\rangle$ as the fixed generator of $H_k(X_k, X_{k-1}; \mathbb{R})$. The same is done for each $\bar{q} \in \mathrm{crit}^+(\bar{f})$.

There exists an $a_{\bar{q}} \in \mathbb{R}$ for each $\bar{q} \in \mathrm{crit}_{k-1}^+(\bar{f})$ such that

$$\partial|\bar{p}\rangle = \sum_{\bar{q} \in \mathrm{crit}_{k-1}^+(\bar{f})} a_{\bar{q}} |\bar{q}\rangle.$$

In order to prove the claim, it suffices to show that

$$a_{\bar{q}} = n(\bar{p}, \bar{q}) = \sum_{\bar{\gamma} \in \mathcal{L}(\bar{p}, \bar{q})} \varepsilon(\bar{\gamma}) \frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|}.$$

In order to do so, we consider smooth singular chains $\partial|\bar{p}\rangle$ and $|\bar{q}\rangle$ in the subspace Y_{k-1} of X, and we utilize the Thom form $\eta_{\bar{q}}$ of $\underline{\mathrm{D}}^+(\bar{q})$ on Y_{k-1} to identify $a_{\bar{q}} = n(\bar{p}, \bar{q})$.

For $\varepsilon < \varepsilon'$, set $\underline{\mathrm{D}}'^-(\bar{p}) = W^-(\bar{p}) \cap \bar{f}^{-1}([k - \varepsilon', \infty))$ and $\partial\underline{\mathrm{D}}'^-(\bar{p}) = \underline{\mathrm{D}}^-(\bar{p}) \cap \bar{f}^{-1}(k - \varepsilon')$. Also, write $|\bar{p}\rangle'$ to denote the singular chain in $\tilde{U}_{\bar{p}}$ representing $D'^-(p)$ over $\underline{\mathrm{D}}'^-(\bar{p})$. The ε' is assumed to be small enough to ensure that there exists a uniformizing chart around $\bar{p}$ such that $\partial\underline{\mathrm{D}}'^{-1}(\bar{p}) \subset \pi_{\bar{p}}(\tilde{U}_{\bar{p}})$.

We define now $\partial|\bar{p}\rangle' = |G_{\bar{p}}|^{-1}\pi_{\bar{p}} \circ \partial|p\rangle'$, and identify $\partial|\bar{p}\rangle$ with $\partial|\bar{p}\rangle'$ by flowing down $\partial|\bar{p}\rangle$ along negative gradient flows from $\bar{f}^{-1}(\varepsilon)$ to $\bar{f}^{-1}(\varepsilon')$.

The formal sum description of $\partial|\bar{p}\rangle$ holds on the homology level, and therefore we can take a formal sum of simplicial complexes K mapping via τ to Y_{k-1}, whose boundary $\partial\tau : \partial K \to Y_{k-1}$ is given by

$$\partial|\bar{p}\rangle' \cup \bigcup_{\bar{q}} a_{\bar{q}}|\bar{q}\rangle,$$

which is endowed with the opposite orientation on the first component relative to X_{k-2}.

We consider the map $\tau : K \to Y_{k-1}$, as well as $\partial|\bar{p}\rangle'$ and $|\bar{q}\rangle$ to all be smooth singular chains on Y_{k-1}. We are allowed rational coefficients since we are over $\mathbb{R}$ in singular homology. Subdividing simplices allows us to assume that τ restricted to each simplex in K has a lift in some chart of $\mathbf{X}$, and so Stokes for orbifolds gives

$$\int_{\partial K} \tau^*\eta_{\bar{q}} = \int_K d(\tau^*\eta_{\bar{q}}) = \int_K \tau^*(d\eta_{\bar{q}}) = 0.$$

It remains to show that $\int_{\partial K} \tau^*\eta_{\bar{q}}$ retrieves $a_{\bar{q}}$. Since $\mathrm{supp}(\eta_{\bar{q}})$ can be shrunken so that it is contained in some small open neighborhood of $\underline{\mathrm{D}}^+(\bar{q})$, we write

$$\int_{\partial K} \tau^*\eta_{\bar{q}} = -\int_I \tau^*\eta_{\bar{q}} + \int_J \tau^*\eta_{\bar{q}} = 0,$$

with I and J subcomplexes of ∂K mapping to $\partial\underline{\mathrm{D}}'^{-1(\bar{p})}$ and $a_{\bar{q}}\underline{\mathrm{D}}'^{-}(\bar{q})$ respectively, via the maps $\partial|\bar{p}\rangle'$ and $a_{\bar{q}}|\bar{q}\rangle$, again respectively.

Observe that $D^+(q) \subset \tilde{U}_{\bar{p}}$ will meet $\partial D'^{-}(p)$ a total of $(\sum_{\bar{\gamma}:\bar{p}\to\bar{q}} \varepsilon(\bar{\gamma})|G_{\bar{p}}|/|G_{\bar{\gamma}}|$ times, considering the orientation of the intersection. This number is the number of gradient flow lines starting at p which lift flow lines in X from $\bar{p}$ to $\bar{q}$.

Take a $G_{\bar{p}}$-invariant differential form $\tilde{\eta}_{\bar{q}}$ to represent the Thom form $\eta_{\bar{q}}$ on $\tilde{U}_{\bar{p}}$. Since on $\tilde{U}_{\bar{p}}$, the integration of a Thom form along a normal fiber of $\tilde{N}$ at $p \in \tilde{N}$ is 1 where $\pi(p) = \bar{p}$, we have that $\eta_{\bar{q}}$ is defined exactly as it is in the smooth case on each uniformizing chart which intersects $\underline{\mathrm{D}}^+(\bar{q})$. So, integrating $\tilde{\eta}_{\bar{q}}$ over $|\bar{p}\rangle$ counts the number of intersection points of $\partial D'^{-}(p)$ and $D^{+}(q)$. This gives that

$$\begin{aligned}\int_I \tau^* \eta_{\bar{q}} &= \frac{1}{|G_{\bar{p}}|} \int_{\partial D'^{-}(p)\tilde{\eta}_{\bar{q}}} \\ &= \frac{1}{|G_{\bar{p}}|} \left(\sum_{\bar{\gamma}:\bar{p}\to\bar{q}} \frac{\varepsilon(\bar{\gamma})|G_{\bar{p}}|}{|G_{\bar{\gamma}}|} \right) \cdot \int_F \tilde{\eta}_{\bar{q}} \\ &= \sum_{\gamma:\bar{p}\to\bar{q}} \frac{\varepsilon(\bar{\gamma})}{|G_{\bar{\gamma}}|},\end{aligned}$$

where F is the general fiber of the normal bundle of $D^{+}(q)$ in $\tilde{U}_{\bar{p}}$. On the other hand, in a uniformizing chart $\tilde{V}_{\bar{q}}$ around $\bar{q}$, we can use the set $D^{-}(q)$ to compute the following.

$$\begin{aligned}\int_J \tau^* \eta_{\bar{q}} &= \frac{a_{\bar{q}}}{|G_{\bar{q}}|} \int_{D^{-}(q)} \tilde{\eta}_{\bar{q}} \\ &= \frac{a_{\bar{q}}}{|G_{\bar{q}}|} \int_{F_q} \tilde{\eta}_{\bar{q}} \\ &= \frac{a_{\bar{q}}}{|G_{\bar{q}}|}.\end{aligned}$$

This result is achieved in part due to the fact that $D^{-}(q)$ and $D^{+}(q)$ meet just once at q. Comparing both integrals and summing them as dictated earlier, we get that

$$a_{\bar{q}} = \sum_{\bar{\gamma}\in\mathcal{L}(\bar{p},\bar{q})} \varepsilon(\bar{\gamma}) \frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|}.$$

As desired, we have shown that the homology of the Morse-Smale-Witten complex for general orbifolds is indeed equal to the singular homology of the underlying quotient space. □

Remark 2.3.3. If we do not have the self-indexing assumption, we can use the following filtration instead, as opposed to the one shown at the beginning of the proof of the preceding theorem.

$$X_k := \bigcup_{\mathrm{ind}(\bar{p})\le k} \underline{\mathrm{W}}^{-}(\bar{p}), \quad \varphi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X.$$

2.4 Density Results for Morse and Morse-Smale Functions on Orbifolds

An important fact about Morse-Smale functions is that they exist and are generic on smooth manifolds [AD13]. That is, Morse functions are dense as an open subset of $C^\infty(M, \mathbb{R})$, meaning that a non-Morse functions can be perturbed slightly to obtain a Morse function; also, it means that given a Morse function (which is guaranteed to exist), there exists a metric g on the manifold such that the transversality condition holds.

In the case of global quotient orbifolds, equivariant Morse functions exist and are dense. Indeed, due to a result of Wasserman [Was69] (and proved later using different techniques by Fukaya et al. [FOOO15], as to be demonstrated), we have the following.

Theorem 2.4.1 (Density of group-invariant Morse functions, [FOOO15][Was69])**.** Suppose M is a manifold on which a finite group G acts effectively (i.e., elements of M are fixed only by 1_G), with a G-invariant Riemannian metric equipped. Denote by $C^k_G(M)$ the set of all G-invariant C^k functions on M. Then the set of all G-invariant, smooth Morse functions on M is a countable intersection of open, dense subsets in $C^\infty_G(M)$.

Proof. Let $K \subset M$ be compact. It suffices to show that the set of all functions in $C^2_G(M)$ which are Morse on K is open and dense. Openness is clear due to the natural topology which we equip C^k-spaces with, the compact-open topology. The rest of what follows is to show density.

We define the following sets in order to stratify our space by the size of isotropy groups at each point. For $p \in X$, consider the isotropy groups $G_p = \{g \in G : gp = p\}$ and define

$$\overset{\circ}{X}(n) = \{p \in X : \#G_p = n\}, \qquad X(n) = \{p \in X : \#G_p \geq n\}.$$

Note that $\overset{\circ}{X}(n)/G$ is a smooth manifold.

First, we claim that if $p \in \overset{\circ}{X}(n)$, then $p \in \operatorname{crit}(f)$ if and only if $p \in \operatorname{crit}(f|_{\overset{\circ}{X}(n)})$. Indeed, this is a consequence of the behavior of the directional derivative, in which $X[f]$ is zero if

$T_pX \ni X \perp \overset{\circ}{X}(n)$. The claim is resolved by the G-invariance of f. $\diamond$

Let us now define the (open) sets

$$A(n) = \{f \in C^\infty_G(V) : \text{all } p \in \text{crit}(f_{X(n)\cap K}) \text{ are Morse}\},$$

$$B(n) = A(n+1) \cap \{f \in C^\infty_G(V) : f|_{\overset{\circ}{X}(n)\cap K} \text{ is Morse}\}.$$

It will suffice to show that $A(1)$ is dense in order to prove the theorem. So, we first claim now that if $A(n+1)$ is dense, then so is $B(n)$.

Proof for $A(n+1) \Longrightarrow B(n)$. Take $W \subset K \cap \overset{\circ}{X}(n)$ to be a relatively compact open subset and define the C^1 map

$$F : W \times C^2_G(\overline{W}) \to T^*\overset{\circ}{X}(n), \quad (x, f) \mapsto D_x f.$$

We have that $F \pitchfork \overset{\circ}{X}(n) \subset T^*\overset{\circ}{X}(n)$, and we can identify $\overset{\circ}{X}(n)$ with the zero section of the bundle $T^*\overset{\circ}{X}(n)$. We define the following set

$$\mathcal{W} = \{(x, f) \in W \times C^2_G(\overline{W}) : F(x, f) \in \overset{\circ}{X}(n) \subset T^*\overset{\circ}{X}(n)\}$$

to be a sub-Banach manifold of the Banach manifold $W \times C^2_G(\overline{W})$. Moreover, the restriction of the projection $\pi : \mathcal{W} \to C^2_G(\overline{W})$ is a Fredholm map (i.e., the kernel and cokernel are finite dimensional), allowing us to apply the Sard-Smale theorem. This gives that the set of regular values of π is dense in $C^2_G(\overline{W})$.

As a subclaim, we say that if f is a regular value of π then $f|_{\overset{\circ}{X}(n)}$ is Morse on W. Indeed, let us take $x \in W$ to be a critical point of f. Then certainly $(x, f) \in \mathcal{W}$. Consider now the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_x\overset{\circ}{X}(n) & \longrightarrow & T_x^*\overset{\circ}{X}(n) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{(x,f)}\mathcal{W} & \longrightarrow & T_x\overset{\circ}{X}(n)\oplus T_fC^2_G(\overline{W}) & \xrightarrow{\overline{D_{(x,f)}F}} & \frac{T_{(x,o)}T^*\overset{\circ}{X}(n)}{T_xT^*\overset{\circ}{X}(n)} = T_n^*\overset{\circ}{X}(n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_fC^2_G(\overline{W}) & \longrightarrow & T_fC^2_G(\overline{W}) & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

For brevity of notation within the diagram, we are denoting by $\overline{D_{(x,f)}F}$ the composition of the induced map $D_{(x,f)}F : T_x\overset{\circ}{X}(n)\oplus T_fC^2_G(\overline{W}) \to T_{(x,o)}T^*\overset{\circ}{X}(n)$ and the projection π. Since f is regular, we have that $T_{(x,f)}\mathcal{W} \to T_fC^2_G(\overline{W})$ is surjective. Diagram chasing yields that $T_x\overset{\circ}{X}(n) \to T_x^*\overset{\circ}{X}(n)$ is surjective as well. This map is equivalently the Hessian at x of $f|_{\overset{\circ}{X}(n)}$, resolving the subclaim. $\diamond$

So, we observe that if $f \in A(n+1)$, then $\mathrm{crit}(f)$ is compact in $\overset{\circ}{X}(n)\cap K$, since it does not have accumulation points on $X(n+1)\cap K$. Equivalently, $B(n)$ is dense. $\square$

Lastly, in order to prove the claim that $A(1)$ is dense, we show that if $B(n)$ is dense, then so is $A(n)$.

Proof for $B(n) \implies A(n)$. Take $f \in B(n)$. Notice that the set of critical points of $f|_{\overset{\circ}{X}(n)}$ on $\overset{\circ}{X}\cap K$ is finite, since $f|_{\overset{\circ}{X}(n)}$ is a Morse function on $\overset{\circ}{X}\cap K$ and doesn't have accumulation points on $X(n+1)\cap K$. Let $\{p_1, ..., p_m\} = \mathrm{crit}(f|_{\overset{\circ}{X}(n)})$ on $\overset{\circ}{X}(n)\cap K$. The Hessian of f at these points are non-degenerate on $T_{p_i}\overset{\circ}{X}(n)$, but could be degenerate on $N_{p_i}\overset{\circ}{X}(n)$. Let us choose functions χ_i and sets V_i such that the following conditions hold:

(i) V_i is a neighborhood of p_i.

(ii) $\mathrm{supp}(\chi_i) \subset V_i$.

(iii) $\chi_i \equiv 1$ in a neighborhood of p_i.

(iv) The $\overline{V}_i$ for $i = 1, ..., m$ are disjoint.

(v) $\overline{V}_i \cap X(n+1) = \emptyset$.

(vi) $gp_i = p_j$ implies $gV_i = V_j$ and $\chi_j \circ g = \chi_i$.

Utilizing the G-invariant Riemannian metric which M is equipped with, we write $f_n(x) = d(x, \mathring{X}(n))^2$. We choose V_i to be sufficiently small such that $\chi_i f_n$ is smooth, satisfying (v). Now a perturbation

$$f_\varepsilon = f + \varepsilon \sum_{i=1}^{m} \chi_i f_n$$

is a Morse function for sufficiently small $\varepsilon > 0$. G-invariance is then given by (vi), and so $f_\varepsilon \in A(n)$. Further, $f_\varepsilon \to f$ as $\varepsilon \to 0$. □

Since the density of $A(n+1)$ implies the density of $B(n)$, which implies the density of $A(n)$, we have that $A(1)$ must be dense. Therefore, we have that the set of smooth, G-invariant Morse functions is indeed dense. □

Although we have density of group-invariant Morse functions as was just shown, we *cannot* guarantee density of Morse-Smale functions [FOOO15]. An example, suggested by Kenji Fukaya, of an orbifold on which Morse-Smale functions are not dense follows.

Example 2.4.1. Consider the manifold $M = \mathbb{R} \times \mathbb{R}^2$, and the group $\mathbb{Z}/n\mathbb{Z}$. Further, if $k \in G$, $x \in \mathbb{R}$, and $w \in \mathbb{R}^2 \cong \mathbb{C}$, define the action (rotation) $[k] \cdot (x, w) = (x, \exp(2\pi i k/n) w)$. Now, define the function $F(x, y, z) = f(x) + g(x)(y^2 + z^2)$ such that

- $f(x)$ has two non-degenerate critical points, with a unique negative gradient flow line ℓ (the red line flowing from p to q) between them, with indices $\lambda_f(p) = 1$ and $\lambda_f(q) = 0$, for example like the function in Figure 2.6.

- $g(x)$ is such that $g(p) > 0$ and $g(q) < 0$. A simple choice is $g(x) = -x$.

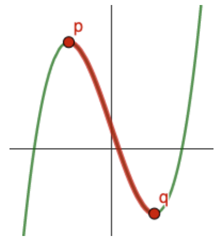

Figure 2.6: *A suitable function $f(x)$*

Certainly then $\nabla F = (f'(x)+g'(x)(y^2+z^2), 2yg(x), 2zg(x))$, and $\text{crit}(F) = \{(x, y, z) : \nabla F = 0\} = \{(p, 0, 0), (q, 0, 0)\}$. Also, we can compute the Hessian matrix (and its restrictions to $\text{crit}(f)$) of F,

$$\text{Hess}(F) = \begin{pmatrix} f''(x) + g''(x)(y^2 + z^2) & 2yg'(x) & 2zg'(x) \\ 2yg'(x) & 2g(x) & 0 \\ 2zg'(x) & 0 & 2g(x) \end{pmatrix} \implies \text{Hess}(F) = \begin{pmatrix} f''(x) & 0 & 0 \\ 0 & -2x & 0 \\ 0 & 0 & -2x \end{pmatrix}$$

Recall now that index is defined to be the number of negative eigenvalues of the Hessian evaluated at a critical point. So, plugging in p yields one negative eigenvalue, whereas q yields 2. That is, $\lambda_F(p, 0, 0) = 1$ and $\lambda_F(q, 0, 0) = 2$. Certainly, the two critical points are non-degenerate, and so F is Morse. However, F is *not* Morse-Smale, as there exists a negative gradient flow line $(\ell, 0, 0)$ from $(p, 0, 0)$ to $(q, 0, 0)$, but we just found that such a trajectory would be going from a lower index point to a higher index point, contradicting an assumption for the Smale condition.

Now, we claim that any small, G-equivariant perturbation of F which we can call $\tilde{F}$ (that is, $\tilde{F}$ is such that $\tilde{F}(g \cdot \xi) = g \cdot \tilde{F}(\xi)$ for all $g \in G$ and $\xi \in \mathbb{R} \times \mathbb{R}^2$. Such a perturbation will have two critical points, $\tilde{p}$ and $\tilde{q}$, which are perturbations of the critical points $(p, 0, 0)$ and $(q, 0, 0)$ of F; these perturbed critical points will have the same index as those of F. However, these perturbed critical points will still be connected by a negative gradient flow line, because when restricted to the fixed locus $\mathbb{R} \times \{0\} \times \{0\}$, the flow ℓ has the correct index and therefore deforms along with $\tilde{F}$. This maintains the issue which we ran into in showing that F is not Morse-Smale, and so it follows that a small perturbation is indeed still not Morse-Smale.

Considering the power of Morse-Smale functions, a future research direction is to study certain cases of orbifolds, and the existence of Morse-Smale functions on them. Indeed, a question worth future study is,

Can one classify orbifolds for which Morse-Smale functions are dense in $C^\infty_G(\mathcal{X}, \mathbb{R})$*?*

In the case of manifolds, it is known that a "bumpy" Riemannian metric gives that a Morse function automatically satisfies the Smale transversality condition. An important question to ask in pursuit of the above question is,

"Does there exist an analogous result for orbifolds, or does the involvement of the G-action cause this to not be the case?"

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