THE CHEEGER-GROMOLL SPLITTING THEOREM

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Contents

1. Introduction

The purpose of this manuscript is to give an exposition on the following theorem, stated now without proof:

Theorem ([CG71])**.** Let (M, g) be a complete Riemannian manifold of nonnegative Ricci curvature. Then (M, g) is isomorphic to the product $\mathbb{R}^k \times N$, where N contains no lines, and $\mathbb{R}^k$ has its usual flat metric.

In other words, if the manifold M contains a certain special geodesic, that is, a line, then we achieve a global product structure; an example of a local-to-global result. This is a surprising result, since a priori, $\mathrm{Ric} \geq 0$ is a relatively loose condition, and the existence of a line feels like a very minimal requirement. However, consider the following example, in which the lack of nonnegative Ricci curvature can be shown to be an obstruction to a product structure.

Example 1.1 (Manifold with line but $\mathrm{Ric} < 0$)**.** Consider the hyperbolic plane $\mathbb{H}$ with its standard metric $\breve{g} = (dx^2 + dy^2)/y^2$. Hyperbolic space is complete and simply connected, and recall that geodesics take the form of semicircles whose centers lie on the x-axis, and vertical rays of the form $x \equiv c$ perpendicular to the x-axis. Let us focus on the latter.

Consider a vertical geodesic $\gamma(t)$ parameterized by arc length, that is, it is unit speed. Such a geodesic follows $x \equiv c$ for some $c \in \mathbb{R}$, hence $dx = 0$. Therefore our metric reduces to the line element $ds^2 = dy^2/y^2$, which we can take the square root of to get $ds = dy/y$ (this is well-defined since $y > 0$ in the upper half plane, so $y = |y|$). Since γ is unit-speed, we have that $ds = |\gamma'|\, dt = dt$. Integrating the line element therefore gives

$$t = \int dt = \int ds = \int \frac{dy}{y} = \ln y + C,$$

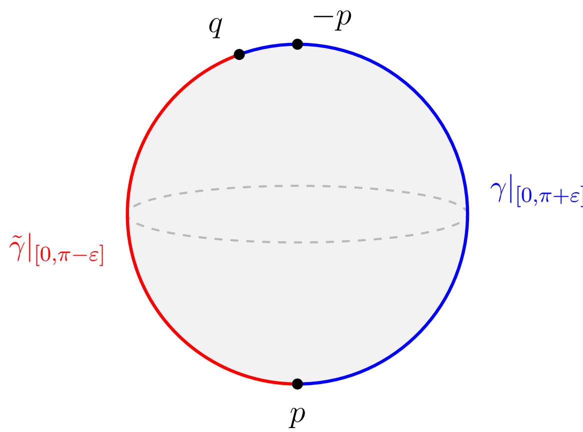

FIGURE 1. There does not exist a globally minimizing geodesic on the sphere S^2: following the blue geodesic γ starting at the point p is minimizing up until the antipode $-p$ of p, after which point it ceases to be minimizing; that is, γ between p and q is not a minimizing geodesic. Instead, it is more efficient to follow the red path $\tilde{\gamma}$, following the other way around the great circle.

and therefore $y(t) = e^t$. Notice that as $t \to -\infty$, $y(t) \to 0$, and as $t \to \infty$, $y(t) \to \infty$. So, $\gamma(t) = (c, e^t)$ is defined for all $t \in \mathbb{R}$. Since $\mathbb{H}$ has constant sectional curvature $\sec = -1$, we can apply the Cartan-Hadamard Theorem (Theorem 12.8 in [Lee18]) to get that $\exp_p : T_pM \to M$ is a global diffeomorphism for all $p \in M$, hence between any two points there exists a unique geodesic; therefore every geodesic segment of γ is minimizing, and so γ is indeed a line.

Since $\sec \equiv -1$, Proposition 8.36 in [Lee18] says $\mathrm{Ric}(v, v) = -g(v, v) < 0$. Suppose then for the sake of contradiction that $\mathbb{H}$ splits. Then we can write $\mathbb{H} = (\mathbb{R} \times N, dt^2 \oplus g_N)$ where N is nontrivial. Suppose ∂_t is the unit vector in the $\mathbb{R}$-direction and X is a unit vector tangent to N, and consider the plane $\Pi = \mathrm{span}\{\partial_t, X\}$. Notice that $\nabla_{\partial_t}\partial_t = \nabla_{\partial_t}X = \nabla_X \partial_t = 0$, so

$$\begin{aligned} R(\partial_t, X)\partial_t &= \nabla_{\partial_t}\nabla_X \partial_t - \nabla_X \nabla_{\partial_t}\partial_t - \nabla_{[\partial_t, X]}\partial_t \\ &= \nabla_{\partial_t} 0 - \nabla_X 0 - \nabla_0 \partial_t \\ &= 0. \end{aligned}$$

Therefore the sectional curvature is

$$\sec(\Pi) = \frac{\langle R(\partial_t, X)X, \partial_t\rangle}{|\partial_t \wedge X|^2} = \frac{\langle 0, \partial_t\rangle}{|\partial_t \wedge X|^2} = 0.$$

This contradicts the fact that $\mathbb{H}$ has constant sectional curvature of -1, so although this manifold contains a line, it does not split.

Similarly, the existence of a *globally* minimal geodesic – a line – also feels like a loose condition; however it can be helpful to consider how lines force noncompactness. Indeed, one can consider the (compact) sphere and see that it is not a product.

Example 1.2 (Manifold with $\mathrm{Ric} \geq 0$ but no line)**.** Consider the unit sphere S^2 with the round metric $\mathring{g}$; then the geodesics are great circles. Notice that for a point $p \in S^2$, its cut locus is the singleton set $\{-p\}$. In particular for $t \in [0, \pi)$, the segment $\gamma|_{[0,t]}$ is minimizing.

At $t = \pi$, we have that $\gamma(p) = -p$, the antipode of p. However, when $t > \pi$, the segment $\gamma|_{[0,t]}$ is not minimizing, since it is now more efficient to go the other way around the great circle to get from p to $\gamma(t)$ – see Figure 1. This result means that not all segments of γ are minimizing (the segments restricted to $[0, t]$ with $t > \pi$), hence there are in fact *no* rays (nor lines, necessarily) on S^2.

Also, the sectional curvature of the sphere is identically 1, and Proposition 8.36 of [Lee18] gives that $\mathrm{Ric}(v, v) = (2 - 1)g(v, v) = g(v, v) > 0$. Notice that S^2 is compact; however if it were to split as $S^2 \cong M \times \mathbb{R}$, this would be a non-compact manifold because of the $\mathbb{R}$-factor.

Lastly, although clearly a product a priori, we can identify nonnegative Ricci curvature and the existence of a line in the following example.

Example 1.3 (Manifold that splits)**.** Consider the manifold $M = S^2_1 \times \mathbb{R}$ with the metric $g = \mathring{g} \oplus dt^2$. Since $\mathrm{Ric}_{M\times N} = \mathrm{Ric}_M \oplus \mathrm{Ric}_N$, we have that $\mathrm{Ric}_{S^2_1\times\mathbb{R}} = \mathrm{Ric}_{S^2_1} + 0 = g_{S^2_1} \geq 0$. Also, if we fix $p \in S^2$, then the curve $\gamma(t) = (p, t)$ is a unit speed geodesic along the $\mathbb{R}$-direction. It is globally minimizing since for any $s < t$, we have $d(\gamma(s), \gamma(t)) = |t - s|$ since we are just moving along the $\mathbb{R}$-direction, hence γ is a line. Certainly this splits.

In this writing milestone, we will showcase the original proof of Cheeger and Gromoll's splitting theorem from [CG71], which proceeds in three main stages: we will construct a (harmonic) Busemann function, then we will show that its gradient is parallel via the Bochner formula, and then we will use de Rham decomposition to obtain a global product structure.

In Section 2, we will build the analytic core of the result by proving some results about Jacobi fields and the index form, from which we will derive the Laplacian comparison theorem given a bound on Ricci curvature. We will then apply these results to the Busemann function associated to a line on the manifold in Section 3, and in doing so, we will use an elliptic regularity argument. In Section 4, we will derive Bochner's formula, which will give that gradient of the Busemann function (restricted to a ray of the line) is a parallel vector field. Section 5 is dedicated to the local de Rham decomposition and its extension to a global result, which we will apply to the parallel vector field found in Section 4 – in order to do so, we introduce the theory of (tangent) distributions and their foliations. This will allow us to complete the proof of the Cheeger-Gromoll theorem in Section 6, and we present consequences of this result in Section 7.

Throughout this document, we write $\square$ to denote the end of a proof, and $\diamond$ the end of a sub-proof in the case of some of the larger theorems that we tackle that contain nontrivial subclaims that don't quite meet the criteria to be separate lemmas. We also denote the space of vector fields on M as $\mathcal{X}(M)$, vector fields along a curve γ as $\mathcal{X}(\gamma)$, and normal vector fields as $\mathcal{X}^\perp(M)$; all vector fields are smooth. $W^{k,p}$ are Sobolev spaces, and $W^{k,2} = H^k$.

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2. Jacobi Fields and Comparison Geometry

In this section, we will recall some of the theory of Jacobi fields and use it to derive a Laplacian comparison estimate for the distance function given a lower bound on Ricci curvature. Jacobi fields encode how geodesics spread, which allows us to convert bounds on curvature into control on volume distortion and the second derivatives of distance. This Laplacian estimate will prove useful in the proof of Cheeger-Gromoll, in which we apply it to a modified distance function, a Busemann function.

2.1. **Variations and Jacobi Fields.**

The underlying constructions behind Jacobi fields are variations of curves. These are extremely useful tools, and will be used throughout this exposition.

Definition 2.1. Let $\gamma : [a,b] \to M$ be a curve. A *variation of* γ is an admissible family of curves $\Gamma : I \times [a,b] \to M$ such that I is an open interval containing 0 and $\Gamma_0 = \gamma$. The variation is called *proper* if $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all s. If each of the main curves $\Gamma_s(t)$ are geodesics, then we say that Γ is a *variation of* γ *through geodesics.* We may omit the words "of γ" when there is no ambiguity.

A variation of a curve $\Gamma(s,t)$ can be thought of as a two-parameter family of curves, where the first parameter s encodes how γ is perturbed, and t acts as the time variable. Note that we refer to $\Gamma_s(t) = \Gamma(s,t)$ as the *main curves* of the variation, and $\Gamma_t(s) = \Gamma(s,t)$ as the *transverse curves* of the variation. In most settings that we care about, the variation will be through geodesics. In this case, we can use the following example as a mental picture of what a variation might look like.

Example 2.2. One of the simplest variations through geodesics that we can construct uses the exponential function. Let $p \in M$ and suppose $v \in T_pM$ is such that $\gamma(t) := \exp_p(tv)$ is a geodesic. We can modify this to form a variation; let $\Gamma(s,t) := \exp_p(t(v + sw))$. Notice that at $s = 0$, $\gamma(t) = \Gamma(0,t) = \exp_p(tv)$.

For each fixed s, the main curve $\Gamma_s(t)$ is a geodesic by definition of the exponential function, with initial point $\Gamma(s,0) = p$ and initial velocity $\partial_t\Gamma(s,0) = v + sw$. Therefore, Γ is a variation through geodesics.

Notice that s perturbs the initial velocity of the geodesic, and t acts as the time flow along the geodesic, as illustrated in Figure 2.

Jacobi fields encode the infinitesimal behavior of variations through geodesics, and therefore measure how nearby geodesics diverge. This is a notion intimately related to curvature, as given by the following definition and proposition: Jacobi fields are exactly the variational vector fields $\partial_s\Gamma$ of variations through geodesics.

Definition 2.3. The *Jacobi equation* along a geodesic γ is $D_t^2 J + R(J, \gamma')\gamma' = 0$, where J is the variation field of a variation through geodesics. If a smooth vector field along a geodesic satisfies this equation, we say it is a *Jacobi field.*

Proposition 2.4. Let $\gamma : \mathbb{R} \to M$ be a geodesic and $V : \mathbb{R} \to TM$ a vector field along γ. Then V satisfies the Jacobi equation if and only if there exists a variation by geodesics

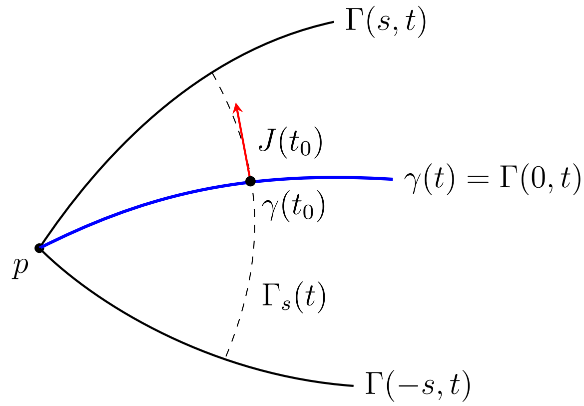

FIGURE 2. A variation $\Gamma(s,t)$ through the geodesic $\gamma(t)$ for some fixed s. The solid lines are main curves $\Gamma_s(t)$, the dashed line is a transverse curve $\Gamma_t(s)$, and the red vector represents the valuation of the Jacobi field J at time t_0.

$\Gamma : (-\varepsilon, \varepsilon) \times \mathbb{R} \to M$ where $V(t) = \partial_s \Gamma_s(t)|_{s=0}$ (that is, $V(t)$ is the velocity of the transverse curve at $s = 0$).

Proof. ($\Longleftarrow$) First write $D_t^2 V = D_t^2(\partial_s \Gamma_s)$. Since the Levi-Civita connection is torsion-free, $D_t^2(\partial_s \Gamma_s) = D_t D_s(\partial_t \Gamma_s)$. Via [Lee18] Proposition 7.5, this is equal to $D_s D_t(\partial_t \Gamma_s) + R(\partial_t \Gamma, \partial_s \Gamma)\partial_t \Gamma$. But, since this is a variation through geodesics, and so Γ_s is a geodesic; hence $D_t(\partial_t \Gamma_s) = 0$ by the geodesic equation. Restricting to $s = 0$, we have that the curvature term becomes $R(\gamma', V)\gamma'$, hence $D_t^2 V = R(\gamma', V)\gamma'$.

($\Longrightarrow$) Suppose that V satisfies the Jacobi equation. Notice that the Jacobi equation is a linear second order ODE, and so there exists a unique solution given initial conditions $J(0)$ and $D_t J(0)$. We want to construct a variation by geodesics, and compare the initial conditions of its variation field with V.

Take $\alpha : (-\varepsilon, \varepsilon) \to M$ to be a curve with $\alpha'(0) = V(0)$, and let $Z : (-\varepsilon, \varepsilon) \to TM$ be a vector field along α with $Z(0) = \gamma'(0)$ and $D_s Z(0) = D_t V(0)$. Then define $\Gamma(s,t) = \exp_{\alpha(s)}(tZ(s))$, and denote $J(t) = \partial_s \Gamma_s(t)|_{s=0}$. Since V satisfies the Jacobi equation, and since Γ is a variation through geodesics, $J(t)$ must also be a Jacobi field via the backward implication we already showed; we then must show that V and J have the same initial conditions.

Indeed, notice

$$J(0) = \partial_s \Gamma(0,0) = \partial_s|_{s=0} \exp_{\alpha(s)}(0) = \alpha'(0) = V(0).$$

Also, since the Levi-Civita connection is torsion-free,

$$D_t J(0) = D_s \partial_t \Gamma(0,0) = D_s \partial_t \exp_{\alpha(s)}(tZ(s))|_{s=t=0} = D_s Z(0) = D_t V(0).$$

Hence the initial conditions align, and we are done. □

This identification is well defined, as Jacobi fields are uniquely determined by their initial conditions. As a result, we will later be able to construct explicit variations through geodesics, and use them to induce information on curvature due to the Jacobi equation.

Proposition 2.5. Suppose $I \subset \mathbb{R}$ is an interval, and $\gamma : I \to M$ is a geodesic, and we have $a \in I$ with $p = \gamma(a)$. Then for every pair of vectors $v, w \in T_pM$, there exists a unique Jacobi field J along γ such that $J(a) = v$ and $D_tJ(a) = w$.

Proof. Take a parallel orthonormal frame $\{E_i\}$ along γ, and write $v = v^iE_i(a)$ and $w = w^iE_i(a)$, and also $\gamma'(t) = y^i(t)E_i(t)$. We can also write an arbitrary vector field $J \in \mathcal{X}(\gamma)$ in coordinates as $J(t) = J^i(t)E_i(t)$, and so the Jacobi equation becomes

$$\ddot{J}^i(t) + R_{jkl}{}^i(\gamma(t))J^j(t)y^k(t)y^l(t) = 0,$$

which is a system of n second order linear ODEs for the n functions $J^i : I \to \mathbb{R}$. We substitute $W^i = \dot{J}^i$ to make it a first order linear system of $2n$ functions:

$$\begin{cases} \dot{J}^i = W^i(t) \\ \dot{W}^i(t) = -R_{jkl}{}^i(\gamma(t))J^j(t)y^k(t)y^l(t). \end{cases}$$

Then there exists a unique smooth solution on I with initial conditions $J^i(a) = v^i$ and $W^i(a) = w^i$. Since $D_tJ(a) = \dot{J}^i(a)E_i(a) = W^i(a)E_i(a) = w$, the desired Jacobi field is $J(t) = J^i(t)E_i(t)$. □

2.2. **Second Variation and the Index Form.** We want to be able to find a geodesic γ such that among all curves connecting two points, γ has minimal length. To do such a thing, we can appeal to the index form $I(V, V)$, which is defined on the space of normal vector fields along a geodesic.

Definition 2.6 (Index form)**.** Let $\gamma : [a, b] \to M$ be a geodesic segment. Then the symmetric bilinear form $I : \mathcal{X}^\perp(\gamma) \times \mathcal{X}^\perp(\gamma) \to \mathbb{R}$ defined by

$$I(V, W) = \int_a^b (\langle D_tV, D_tW\rangle - \mathrm{Rm}(V, \gamma', \gamma', W))\, dt$$

is called the *index form*.

The index form is a quadratic form that can be used to show that a geodesic γ defined on $[0, r]$ is length minimizing by measuring how the length of a geodesic changes under nearby variations. This follows from the following proposition, showing the connection between length minimization and curvature.

Proposition 2.7. Let γ be a unit speed geodesic segment on $[0, r]$, and let Γ be a proper variation of γ. Then $I(V, V) = \frac{d^2}{ds^2}|_{s=0}L_g(\Gamma_s)$ for $V \in \mathcal{X}(\gamma)$.

Proof. By definition of the second variation,

$$\frac{d^2}{ds^2}\bigg|_{s=0} L_g(\Gamma_s) = \int_0^r \left(|D_tV^\perp|^2 - \mathrm{Rm}(V^\perp, \gamma', \gamma', V^\perp)\right)\, dt,$$

and for normal vector fields $V, W \in \mathcal{X}^\perp(\gamma)$, we have

$$I(V, W) = \int_0^r (\langle D_tV, D_tW\rangle - \mathrm{Rm}(V, \gamma', \gamma', W))\, dt.$$

Surely then for $V = W$ (taken to be normal), the two expressions are equal. □

The reason we care only about normal vector fields is because the geometry behind the spreading or convergence of geodesics comes only from the directions normal to the geodesic, and the Hessian of r is only nonzero on vectors orthogonal to ∇r, which we will see later. That is to say, the comparison theorem that we build up to in this chapter is inherently a statement on the directions transverse to ∇r, which by the first variation formula coincides with γ', which we show in Lemma 2.17. Further, a vector field with a tangential component would decompose as $V = V^{\perp} + \langle V, \gamma' \rangle \gamma'$, and the tangential terms would create extra information that is unnecessary, complicating the typical assumption that γ is a unit speed geodesic.

We now give various properties of the index form as it relates to Jacobi fields and proper vector fields. Throughout, take γ to be a geodesic.

Proposition 2.8. The index form is bilinear and symmetric.

Proof. For symmetry, fix some geodesic γ and take $V, W \in \mathcal{X}^{\perp}(\gamma)$. We have the following identity due to the symmetries of the curvature tensor:

$$\operatorname{Rm}(V, \gamma', \gamma', W) = -\operatorname{Rm}(\gamma', V, \gamma', W) = \operatorname{Rm}(\gamma', V, W, \gamma') = \operatorname{Rm}(W, \gamma', \gamma', V).$$

By this identity and the symmetry of inner products,

$$\begin{aligned} I(V, W) &= \int_0^r \left(\langle D_t V, D_t W \rangle - \operatorname{Rm}(V, \gamma', \gamma', W) \right) dt \\ &= \int_0^r \left(\langle D_t W, D_t V \rangle - \operatorname{Rm}(W, \gamma', \gamma', V) \right) dt \\ &= I(W, V). \end{aligned}$$

For bilinearity, we check that $I(aV_1 + bV_2, W) = aI(V_1, W) + bI(V_2, W)$. This follows from the linearity of the covariant derivative along a curve, as well as the C^{∞}-multilinearity of the Riemann curvature tensor:

$$\begin{aligned} I(aV_1 + bV_2, W) &= \int_0^r \langle D_t(aV_1 + bV_2), D_t W \rangle + \operatorname{Rm}(aV_1 + bV_2, \gamma', \gamma', W)\, dt \\ &= \int_0^r \langle aD_t(V_1) + bD_t(V_2), D_t W \rangle + a \operatorname{Rm}(V_1, \gamma', \gamma', W) + b \operatorname{Rm}(V_2, \gamma', \gamma', W)\, dt \\ &= \int_0^r a\langle D_t V_1, D_t W \rangle + b\langle D_t V_2, D_t W \rangle + a \operatorname{Rm}(V_1, \gamma', \gamma', W) + b \operatorname{Rm}(V_2, \gamma', \gamma', W)\, dt \\ &= \int_0^r a\langle D_t V_1, D_t W \rangle + a \operatorname{Rm}(V_1, \gamma', \gamma', W)\, dt \\ &\qquad + \int_0^r b\langle D_t V_2, D_t W \rangle + b \operatorname{Rm}(V_2, \gamma', \gamma', W)\, dt \\ &= aI(V_1, W) + bI(V_2, W). \end{aligned}$$

Showing linearity in the second slot follows identically as the first. ∎

We can consider the index form I as a quadratic functional defined on $X^{\perp}(\gamma)$, mapping $V \mapsto I(V, V)$. We say that $V \in X^{\perp}(\gamma)$ is a critical point of $I(V, V)$ if its first variation vanishes in every admissible direction $W \in X^{\perp}(\gamma)$, that is $\frac{d}{d\varepsilon}|_{\varepsilon=0} I(V+\varepsilon W, V+\varepsilon W) = 0$. The following proposition gives an alternate condition.

Proposition 2.9. $V \in \mathcal{X}^{\perp}(\gamma)$ is a critical point of $I(V,V)$ if and only if $I(V,W) = 0$ for all admissible $W \in \mathcal{X}^{\perp}(\gamma)$ with the same endpoints.

Proof. ($\Longrightarrow$) Suppose that V is a critical point of I. Then $\frac{d}{d\varepsilon}|_{\varepsilon=0} I(V+\varepsilon W, V+\varepsilon W) = 0$ for all admissible W. We can expand the left hand side as

$$\begin{aligned} I(V+\varepsilon W, V+\varepsilon W) &= I(V,V) + \varepsilon I(W,V) + \varepsilon I(V,W) + \varepsilon^2 I(W,W) \\ &= I(V,V) + \varepsilon(I(W,V) + I(V,W)) + \varepsilon^2 I(W,W) \\ &= I(V,V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W), \end{aligned}$$

where the last equality comes from Proposition 2.8. Differentiating with respect to ε gives

$$\frac{d}{d\varepsilon} I(V+\varepsilon W, V+\varepsilon W) = 2I(V,W) + 2\varepsilon I(W,W),$$

and evaluation at zero gives $\frac{d}{d\varepsilon}|_{\varepsilon=0} I(V+\varepsilon W, V+\varepsilon W) = 2I(V,W)$, and by assumption this is equal to 0 when V is a critical point. Certainly then if $2I(V,W) = 0$, then $I(V,W) = 0$.

($\Longleftarrow$) Next, fix W and suppose that $I(V,W) = 0$. Since $\frac{d}{d\varepsilon}|_{\varepsilon=0} I(V+\varepsilon W, V+\varepsilon W) = 2I(V,W)$, this assumption gives that the derivative is zero, hence V is a critical point. □

The following lemma gives an alternate expression for the index form, which comes in handy in some of the following propositions. By writing $I(V,W)$ with a $D_t^2 V$ term, it will be much easier to substitute curvature terms in accordance to the Jacobi equation into the index form.

Lemma 2.10. Let $\gamma : [a,b] \to M$ be a geodesic segment. Then for every pair of smooth vector fields V and W along γ,

$$I(V,W) = \langle D_t V, W \rangle \Big|_a^b - \int_a^b \langle D_t^2 V + R(V,\gamma')\gamma', W \rangle \, dt.$$

Proof. By the product rule, $\frac{d}{dt}\langle D_t V, W \rangle = \langle D_t^2 V, W \rangle + \langle D_t V, D_t W \rangle$. After rearranging, the fundamental theorem of calculus gives

$$\int_a^b \langle D_t V, D_t W \rangle \, dt = \langle D_t V, W \rangle \Big|_a^b - \int_a^b \langle D_t^2 V, W \rangle \, dt.$$

□

Proposition 2.11. $I(J,W) = 0$ for all admissible, normal, proper $W \in \mathcal{X}^{\perp}(\gamma)$ if and only if J is a Jacobi field.

Proof. ($\Longleftarrow$) Suppose J is Jacobi. Then integrating the first term in the index form by parts as in Lemma 2.10, we see

$$\int_0^r \langle D_t J, D_t W \rangle \, dt = \langle D_t J, W \rangle \Big|_0^r - \int_0^r \langle D_t^2 J, W \rangle \, dt.$$

Since W is proper, the first term vanishes since $W(0) = W(r) = 0$. By linearity and by the Jacobi equation,

$$\begin{aligned} I(J,W) &= -\int_0^r \langle D_t^2 J, W\rangle\,dt - \int_0^r \mathrm{Rm}(J,\gamma',\gamma',W)\,dt \\ &= -\int_0^r \langle D_t^2 J, W\rangle\,dt - \int_0^r \langle R(J,\gamma')\gamma', W\rangle\,dt \\ &= -\int_0^r \langle D_t^2 J + R(J,\gamma')\gamma', W\rangle\,dt \\ &= -\int_0^r \langle 0, W\rangle\,dt \\ &= 0. \end{aligned}$$

($\implies$) Conversely, suppose that $I(J,W) = 0$ for arbitrary W. Then

$$I(J,W) = -\int_0^r \langle D_t^2 J + R(J,\gamma')\gamma', W\rangle\,dt = 0.$$

But if this integral is 0, this means that the inner product must be 0, but since W is arbitrary, we must have $D_t^2 J + R(J,\gamma')\gamma' = 0$, hence J is Jacobi. □

Proposition 2.12. Take $p, q \in M$, and γ to be a unit speed geodesic segment on $[p,q]$ that has an interior conjugate point. Then there exists a proper $V \in \mathcal{X}^\perp(\gamma)$ such that $I(V,V) < 0$.

Proof. Suppose $\gamma : [a,c] \to M$ is a unit speed geodesic segment, and $\gamma(b)$ is conjugate to $\gamma(a)$ along γ for some $b \in (a,c)$. Therefore there exists a nontrivial normal Jacobi field J along γ that vanishes at $t = a$ and $t = b$. If we define a vector field V along γ as

$$V(t) = \begin{cases} J(t) & t \in [a,b], \\ 0 & t \in [b,c], \end{cases}$$

then this is a proper, normal, piecewise smooth vector field along γ. Notice that since this is just piecewise smooth, the derivative $D_t V$ has a discontinuity at $t = b$. In particular, define the "jump" of $D_t V$ at $t = b$ to be

$$\delta D_t V := D_t V(b^+) - D_t V(b^-) = 0 - D_t J(b) = -D_t J(b).$$

This must be nonzero, because otherwise J would be a Jacobi field satisfying $J(b) = D_t J(b) = 0$, and thus would be identically 0.

Now suppose W is a smooth proper normal vector field along γ such that $W(b)$ is equal to $\delta D_t V$ at $t = b$. We can construct such a W by taking a smooth orthonormal frame $(E_1(t), ..., E_{n-1}(t), \gamma'(t))$ along γ by parallel transporting the basis of $T_{\gamma(a)}M$ along γ, and writing the vector

$$-D_t J(b) = \sum_{i=1}^{n-1} \langle -D_t J(b), E_i(b)\rangle \cdot E_i(b).$$

To smooth this out, consider the smooth bump function $\varphi : [a, c] \to [0, 1]$ defined by $\varphi(a) = 0$, $\varphi(b) = 1$, and $\varphi(c) = 0$, and then we can define

$$W(t) = \varphi(t) \sum_{i=1}^{n-1} \langle -D_t J(b), E_i(b) \rangle \cdot E_i(t).$$

For a small $\varepsilon > 0$, let $X_\varepsilon = V + \varepsilon W$. Then

$$I(X_\varepsilon, X_\varepsilon) = I(V + \varepsilon W, V + \varepsilon W) = I(V, V) + 2\varepsilon I(V, W) + \varepsilon^2 I(W, W).$$

Since V satisfies the Jacobi equation on both $[a, b]$ and $[b, c]$, and since $V(b) = 0$, Lemma 2.10 gives that $I(V, V) = -\langle \delta D_t V, V(b) \rangle = 0$ and $I(V, W) = -\langle \delta D_t V, W(b) \rangle = -|W(b)|^2$. Therefore

$$I(X_\varepsilon, X_\varepsilon) = -2\varepsilon |W(b)|^2 + \varepsilon^2 I(W, W) = \varepsilon(-2|W(b)|^2 + \varepsilon I(W, W)).$$

Taking ε small enough gives that this is strictly negative as desired. □

We have established that a Jacobi field J is both a critical point of the index form, and also that the index form of a proper Jacobi field is 0. If we consider the additional requirement that our geodesic contains no conjugate points (which is a logical constraint to make, considering in the context of this document: the geodesic of study is a line), then this means that the index form is in fact *minimized* when its input is a Jacobi field, due to the following.

Theorem 2.13. Let $\gamma : [0, r] \to M$ be a unit speed geodesic segment without conjugate points, and let V be any normal smooth vector field along γ with endpoints $V(0) = 0$ and $V(r) = v \in \gamma'(r)^\perp$. If J is the unique (Proposition 2.5) normal Jacobi field along γ with the same endpoints as V, then $I(J, J) \le I(V, V)$ for all $V \in \mathcal{X}^\perp(\gamma)$, where equality holds if and only if V is Jacobi.

Proof. We are claiming that for every proper $V \in \mathcal{X}^\perp(\gamma)$, where γ contains no conjugate points, $I(V, V) \ge I(J, J)$ for J a proper Jacobi field, with equality if and only if $V \equiv J$. Fix V, and set $W = V - J$. Since V and J have the same endpoints, $W(0) = V(0) - J(0) = 0$ and $W(r) = V(r) - J(r) = 0$, hence W is proper. By Proposition 2.8,

$$I(V, V) = I(J + W, J + W) = I(J, J) + 2I(J, W) + I(W, W).$$

To show the claim, we want to show that $I(J, W) = 0$ and $I(W, W) \ge 0$ for proper W.

To show $I(J, W) = 0$, we differentiate as follows:

$$\frac{d}{dt} \langle D_t J, W \rangle = \langle D_t^2 J, W \rangle + \langle D_t J, D_t W \rangle.$$

Integrating the second term,

$$\int_0^r \langle D_t J, D_t W \rangle \, dt = \langle D_t J, W \rangle \Big|_0^r - \int_0^r \langle D_t^2 J, W \rangle \, dt.$$

The first term is 0 since W is proper, and therefore

$$I(J, W) = \int_0^r \langle D_t J, D_t W \rangle - R(J, \gamma', \gamma', W) \, dt$$

$$= \int_0^r \langle D_t J, D_t W\rangle \, dt - \int_0^r \langle R(J,\gamma')\gamma', W\rangle \, dt$$
$$= -\int_0^r \langle D_t^2 J, W\rangle - \int_0^r \langle R(J,\gamma')\gamma', W\rangle \, dt$$
$$= -\int_0^r \langle D_t^2 J - R(J,\gamma')\gamma', W\rangle \, dt.$$

Since J is Jacobi, the inner product is 0. Therefore $I(V,V) = I(J,J) + I(W,W)$.

By Proposition 2.12, $I(W,W) < 0$ if there exists a conjugate point along the geodesic. But γ has no conjugate points and so $I(W,W) \geq 0$ by contraposition. Hence $I(V,V) = I(J,J) + I(W,W) \geq I(J,J)$. □

2.3. **Laplacian Comparison for Nonnegative Ricci Curvature.** As we have seen, the index form provides a way to connect information about vector fields (more specifically, Jacobi fields) to the Riemann curvature tensor. To get towards the heart of the Cheeger-Gromoll theorem, we want to relate the index form to Ricci curvature. In this section, we will make this connection, and also see how the index form of Jacobi fields relates to the Laplacian of the distance function. Namely, the remainder of Section 2 is dedicated to proving Theorem 2.18.

Definition 2.14. For each $c \in \mathbb{R}$, let us define a function $s_c : \mathbb{R} \to \mathbb{R}$ by

$$s_c(t) = \begin{cases} t & \text{if } c = 0, \\ R\sin(t/R) & \text{if } c = 1/R^2 > 0, \\ R\sinh(t/R) & \text{if } c = -1/R^2 < 0. \end{cases}$$

We interpret c as being the constant sectional curvature of some space. For the purposes of the comparison theorem that we need to show the Cheeger-Gromoll splitting theorem, we want to compare against Euclidean space, which has constant curvature $c = 0$. This function then allows for Jacobi fields to be written as follows.

Proposition 2.15. If M has constant sectional curvature c, and γ is a unit speed geodesic, then the normal Jacobi fields along γ that vanish at $t = 0$ are of the form $J(t) = ks_c(t)E(t)$, where E is a parallel unit normal vector field along γ and k is an arbitrary constant.

Corollary 2.16. In a manifold of constant sectional curvature c, the Jacobi equation is $(ks_c)'' + c(ks_c) = 0$.

Proof. By the above proposition, we have that $J(t) = ks_c(t)E(t)$. Since E is a parallel unit normal vector field, $D_t^2 J(t) = (ks_c(t))''E(t)$. Also by Proposition 8.36 in [Lee18],

$$R(ks_cE,\gamma')\gamma' = c(\langle\gamma',\gamma'\rangle ks_cE - \langle ks_cE,\gamma'\rangle\gamma') = cks_cE.$$

Therefore the Jacobi equation becomes $(ks_c)'' + c(ks_c) = 0$. □

The following lemma is the remaining piece needed to prove the desired Laplacian comparison theorem, which immediately follows.

Lemma 2.17. Let $r(x) = d(p, x)$ be the radial distance function in a normal neighborhood of the point p, and let $\gamma : [0, r] \to M$ be the unique unit speed minimizing geodesic from p to x with $\gamma(0) = p$ and $\gamma(r) = x$. Then $\nabla r(x) = \gamma'(r)$.

Proof. By definition of the gradient, we want to show that $dr_x(v) = \langle \gamma'(r), v\rangle$ for all $v \in T_pM$. Fix $v \in T_xM$, and let $\sigma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\sigma(0) = x$ and $\sigma'(0) = v$. Then we can define $f(s) = r(\sigma(s)) = d(p, \sigma(s))$, so that by construction, $dr_x(v) = f'(0)$.

Consider now the variation through geodesics $\Gamma(s, t) = \gamma_s(t)$ with $0 \leq t \leq f(s)$, where each γ_s is a unit speed geodesic. Let $T = \partial_t\Gamma$ and $S = \partial_s\Gamma$; then $|T| = 1$ and $\Gamma(s, f(s)) = \sigma(s)$. The first variation formula for length gives

$$\frac{d}{ds}L(\gamma_s)\bigg|_{s=0} = \langle S(0, r), T(0, r)\rangle - \langle S(0, 0), T(0, 0)\rangle + \int_0^r \langle \nabla_t T, S\rangle \, dt.$$

Since $L(\gamma_s) = f(s)$, the left hand side is $f'(0)$. Since the γ_s are geodesics, $\nabla_t T = 0$, so the integral term is 0. Since the base point is fixed, $\Gamma(s, 0) = p$ for all s, and so $S(0, 0) = 0$, so the second term is 0. At the endpoint, $\Gamma(s, f(s)) = \sigma(s)$, and so differentiating at $s = 0$ gives $S(0, r) = \sigma'(0) = v$. Since $\gamma_0 = \gamma$, we also have that $T(0, r) = \gamma'(r)$. Therefore the first variation formula reduces to $f'(0) = \langle v, \gamma'(r)\rangle$. But $f'(0) = dr_x(v)$, so we are done. □

Theorem 2.18 (Laplacian Comparison)**.** Suppose $\mathrm{Ric}(v, v) \geq (n-1)c$ for some $c \in \mathbb{R}$ for all unit vectors v, and suppose M contains a line. Given any $p \in M$, take U to be a normal neighborhood of p, and let r be the radial distance function from p on U. Then $\Delta r \leq (n-1)s_c'(r)/s_c(r)$ holds on $U \setminus \{p\}$.

Proof. Let γ be a line. Then it is globally minimizing, and so it has no conjugate points on $[0, r]$ for any r. Therefore $I(W, W) \geq 0$ for any proper $W \in \mathcal{X}^\perp(\gamma)$ by Theorem 2.13. We can pick an orthonormal frame $\{E_1(0), ..., E_{n-1}(0)\} \subset \dot\gamma(0)^\perp$, and extend this via parallel transport to get a frame $\{E_i(t)\}$ for each t. Necessarily, $D_tE_i = 0$ since the E_i are parallel, and $E_i(t) \perp \dot\gamma$. With this, define the vector fields $W_i(t) = \varphi(t)E_i(t)$, where φ is a scalar function such that $\varphi(0) = 0$ and $\varphi(r) = 1$. The W_i are normal by construction, and are such that $W_i(0) = 0$ and $W_i(r) = E_i(r)$.

Since the E_i are parallel, $D_tW_i = \varphi'(t)E_i(t)$, and by orthonormality, $|D_tW_i|^2 = (\varphi')^2$. By linearity of the curvature endomorphism, $\langle R(W_i, \gamma')\gamma', W_i\rangle = \varphi^2\langle R(E_i, \gamma')\gamma', E_i\rangle$. Therefore we can write the index form as

$$I(W_i, W_i) = \int_0^r \left((\varphi')^2 - \varphi^2\langle R(E_i, \gamma')\gamma', E_i\rangle\right) \, dt,$$

and summing over i, we see the appearance of Ricci curvature:

$$\begin{aligned}\sum_{i=1}^{n-1} I(W_i, W_i) &= \int_0^r \left(\sum_{i=1}^{n-1}(\varphi')^2 - \sum_{i=1}^{n-1}\varphi^2\langle R(E_i, \gamma')\gamma', E_i\rangle\right) \, dt \\ &= \int_0^r \left((n-1)(\varphi')^2 - \varphi^2 \,\mathrm{Ric}(\gamma', \gamma')\right) \, dt.\end{aligned}$$

Since each $I(W_i, W_i)$ is nonnegative by Proposition 2.12 and Theorem 2.13, this expression is nonnegative. For a stronger bound that will eventually yield the Laplacian, we can bound the $I(W_i, W_i)$ below by the index form of the unique Jacobi field J_i with the same endpoints as W_i by Theorem 2.13, and if we introduce the bound $\text{Ric} \geq (n-1)c$ (where in our context we take $c = 0$ to get the hypothesis $\text{Ric} \geq 0$), we get

$$0 \leq \sum_{i=1}^{n-1} I(J_i, J_i) \leq \sum_{i=1}^{n-1} I(W_i, W_i) \leq (n-1) \int_0^r \left((\varphi')^2 - c\varphi^2\right) dt.$$

To evaluate this integral in a meaningful way, we want to assign a value to φ. To do so, consider the function s_c as in Definition 2.14. Pick $\varphi(t) = s_c(t)/s_c(r)$ (that is, $\varphi(t) = ks_c(t)$ with $k = 1/s_c(r)$), or in the (Euclidean) case that we will care about, we can specify to $\varphi(t) = t/r$. Therefore $\varphi(0) = 0$ and $\varphi(r) = 1$, satisfying the properties we used to define the W_i. Now, fix a vector $v \in \dot{\gamma}(r)^\perp$, and take J to be the unique Jacobi field with $J(0) = 0$ and $J(r) = v$. Among all vector fields W with $W(0) = 0$ and $W(r) = v$, the Jacobi field minimizes the index form as shown in Theorem 2.13 for the case that we have no conjugate points since we are taking γ to be a line.

By Corollary 2.16, $\varphi'' + c\varphi = 0$, hence $\varphi'' = -c\varphi$. Then

$$\frac{d}{dt}(\varphi\varphi') = (\varphi')^2 + \varphi\varphi'' = (\varphi')^2 - c\varphi^2.$$

Integrating, we have

$$\int_0^r ((\varphi')^2 - c\varphi^2)\, dt = [\varphi\varphi']_0^r = \varphi(r)\varphi'(r) - \varphi(0)\varphi'(0) = \varphi'(r) = \frac{s_c'(r)}{s_c(r)},$$

since $\varphi(0) = 0$ and $\varphi(r) = 1$ by definition. So, $\sum_{i=1}^{n-1} I(J_i, J_i) \leq (n-1)s_c'(r)/s_c(r)$.

It remains to show that $\Delta r = \sum_{i=1}^{n-1} I(J_i, J_i)$. Since γ is a line, Jacobi fields are not proper on any segment $\gamma|_{[0,r]}$. In particular, we have the boundary conditions $J_i(0) = 0$ and $J_i(r) = E_i(r)$ on the segment $\gamma|_{[0,r]}$.

We want to build a variation whose variation field is J_i. Away from the cut locus of $\gamma(0)$, the exponential map is a diffeomorphism in a neighborhood of the vector $r\gamma'(0)$. In particular, we can choose initial velocities $v(0) = \gamma'(0)$ and $v(s) \in T_pM$ with $|v(s)| = 1$ such that $x(s) := \exp_p(rv(s))$ has initial conditions $x(0) = \gamma(r)$ and $x'(0) = E_i(r)$. Then we can define the variation $\Gamma^i(s,t) := \exp_p(tv(s))$ for $t \in [0, r]$. Γ^i is such that $\Gamma^i_s(t)$ is a unit speed geodesic from $\gamma(0)$ to $x(s)$, $\Gamma^i_0(t) = \gamma(t)$, and the variation field is $J_i(t) := \partial_s\Gamma^i(t)|_{s=0}$; this J_i is a Jacobi field by Proposition 2.4, and it has endpoints $J_i(0) = 0$ and $J_i(r) = x'(0) = E_i(r)$.

Along $\gamma|_{(0,r]}$ we have $\nabla r(\gamma(r)) = \gamma'(r)$ by Lemma 2.17. By definition of the Hessian, $\text{Hess}\, f(V, V) = \langle \nabla_V(\nabla f), V\rangle$. Applying this to the radial distance function along the vector field $E_i(r)$,

$$\text{Hess}\, r(E_i(r), E_i(r)) = \langle \nabla_{E_i(r)}(\nabla r), E_i(r)\rangle = \langle \nabla_{E_i(r)}\gamma', E_i(r)\rangle.$$

Define the vector fields $T^i = \partial\Gamma^i/\partial t$ and $S^i = \partial\Gamma^i/\partial s$. Evaluating at $(s,t) = (0, r)$ gives $T^i(0,r) = \gamma'(r)$ and $S^i(0,r) = J_i(r) = E_i(r)$, and also $\nabla_{T^i} S^i|_{s=0} = D_t J_i(r)$ by definition. Since the Levi-Civita connection is torsion-free, $\nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_s$, hence $\nabla_{S^i(0,r)} T^i(0,r) = \nabla_{T^i(0,r)} S^i(0,r)$, equivalently $\nabla_{J_i(r)}\gamma'(r) = D_t J_i(r)$. Thus $\nabla_{E_i(r)}\gamma'(r) = D_t J_i(r)$ since $J_i(r) =$

$E_i(r)$. Therefore,

$$\operatorname{Hess} r(E_i(r), E_i(r)) = \langle \nabla_{E_i(r)}\gamma', E_i(r)\rangle = \langle D_t J_i(r), E_i(r)\rangle.$$

This gives the desired result. Indeed, recall that $\frac{d}{dt}\langle X, Y\rangle = \langle D_t X, Y\rangle + \langle X, D_t Y\rangle$. Hence $\frac{d}{dt}\langle D_t J_i, J_i\rangle = |D_t J_i|^2 + \langle D_t^2 J_i, J_i\rangle$. Since J_i is Jacobi, $D_t^2 J_i = -R(J_i, \gamma')\gamma'$, and so

$$\begin{aligned} I(J_i, J_i) &= \int_0^r |D_t J_i|^2 - \langle R(J_i, \gamma')\gamma', J_i\rangle \, dt \\ &= \int_0^r |D_t J_i|^2 + \langle D_t^2 J_i, J_i\rangle \, dt \\ &= \int_0^r \frac{d}{dt}\langle D_t J_i, J_i\rangle \, dt \\ &= \langle D_t J_i(r), J_i(r)\rangle - \langle D_t J_i(0), J_i(0)\rangle \\ &= \langle D_t J_i(r), J_i(r)\rangle \\ &= \langle D_t J_i(r), E_i(r)\rangle. \end{aligned}$$

Finally, we see $\operatorname{Hess} r(E_i(r), E_i(r)) = I(J_i, J_i)$. Taking a sum, we achieve the comparison:

$$\Delta r = \sum_{i=1}^{n-1} \operatorname{Hess} r(E_i(r), E_i(r)) = \sum_{i=1}^{n-1} I(J_i, J_i) \le (n-1)\frac{s_c'(r)}{s_c(r)}.$$

□

3. Busemann Functions and Elliptic Regularity

3.1. **Rays and Lines.** Suppose M is a complete Riemannian manifold. Recall that *complete* in this sense has dual meanings, as given by the Hopf-Rinow Theorem: M is geodesically complete (i.e., every maximal geodesic is defined on all of $\mathbb{R}$), and M is complete as a metric space under the Riemannian distance function, and these two "completenesses" coincide.

Definition 3.1. A *ray* (resp. *line*) in M is a geodesic $\gamma : [0, \infty) \to M$ (resp. $\gamma : \mathbb{R} \to M$) such that each segment of γ is minimal.

The condition regarding minimality feels clear, however the following example will demonstrate how this is in fact a rather restrictive condition, and is closely tied to the idea of conjugate points and cut loci.

Example 3.2. Consider the infinite cylinder $S^1 \times \mathbb{R}$, which has geodesics defined along the $\mathbb{R}$-direction, as well as geodesics that wrap around S^1 (viewed projectively) and are helix shaped, with "tightness" of the coil determined by the amount of the $\mathbb{R}$ component in the initial velocity vector. Recall that geodesics are locally length minimizing; lines, however, are geodesics that are *globally* minimizing. Hence the only lines in the cylinder are those that have no S^1 component, that is, they can be expressed as $\gamma_\theta(x) = tx$ as t varies in $\mathbb{R}$. To see how this fails for other (helicoid) geodesics, consider a helix passing through points $p = (\theta, z_1)$ and $q = (\theta, z_2)$. Certainly this helix is not length minimizing, since we can take the vertical line along the angle θ.

Another picture to have in mind is Figure 1 and Example 1.2 from the introduction. Notice too the important fact about how rays and lines have a strong impact on the global topology of the manifold.

Proposition 3.3. A ray exists on a complete manifold M if and only if M is not compact.

Proof. ($\implies$) Suppose that $\gamma : [0, \infty) \to M$ is a ray, and for the sake of contradiction, suppose M is compact. Then $\gamma([0, \infty)) \subset M$ is an infinite subset of a compact space, and so there exist $\{t_n\}$ tending to ∞ as $n \to \infty$ such that $\{\gamma(t_n)\}$ contains a subsequence $\{\gamma(t_{n_k})\}$ converging to some $p \in M$ as $k \to \infty$. In particular, $d(\gamma(t_{n_k}), \gamma(t_{n_\ell})) \to 0$ as $k, \ell \to \infty$.

However, since γ is a ray, we have that $d(\gamma(t_{n_k}), \gamma(t_{n_\ell})) = |t_{n_k} - t_{n_\ell}|$. Since the sequence $\{t_{n_k}\}$ is unbounded, we can choose k, ℓ such that $|t_{n_k} - t_{n_\ell}| > \varepsilon$, giving a contradiction. Hence M must be noncompact.

($\impliedby$) Fix $p \in M$. Since M is not compact, there does not exist a closed ball that contains M; that is, for each $k \in \mathbb{N}$, there exists a point $q_k \in M$ such that $d(p, q_k) \geq k$. Also since M is complete, the closed and bounded sets are compact, and so for each k, $\overline{B}_k(p)$ is compact.

Consider the function $f(x) = d(p, x)$ on $\overline{B}_k(p)$. Since the ball is compact, f attains its maximum at some $x_k \in \overline{B}_k(p)$, hence $d(p, x_k) = k$. By Hopf-Rinow, there exists a minimizing unit speed geodesic $\gamma_k : [0, k] \to M$ from p to x_k, that is, $\gamma_k(0) = p$ and $\gamma_k(k) = x_k$, and for all $s < t$ between 0 and k, we have $d(\gamma_k(s), \gamma_k(t)) = t - s$.

Define the initial unit tangent vectors $\{v_k\} = \{\gamma_k'(0)\} \subset S_pM$. Since S_pM is compact, there exists a subsequence $\{v_{k_j}\} \subset S_pM$ and a unit vector v such that $v_{k_j} \to v \in S_pM \subset T_pM$. If we define $\gamma : [0, \infty) \to M$ to be the geodesic $\gamma(t) = \exp_p(tv)$ (which is defined for all $t \geq 0$ by completeness), then this is the ray that we are looking for. Indeed, if we fix $s < t$, then for all $k \geq t$, we have that γ_k is minimizing on $[s, t]$. Therefore $d(\gamma_k(s), \gamma_k(t)) = t - s$. Since $[0, t]$ is compact, $\gamma_k \to \gamma$ uniformly, and so $\gamma_k(s) \to \gamma(s)$ and $\gamma_k(t) \to \gamma(t)$. Therefore $d(\gamma(s), \gamma(t)) = \lim_{k\to\infty} d(\gamma_k(s), \gamma_k(t)) = t - s$ as desired. □

3.2. **Busemann Functions.** Given a ray $\gamma : [0, \infty) \to M$, we would like a way to measure how far a point $x \in M$ is from "infinity" along γ. The object that will allow us to do this is the Busemann function, the main geometric object of interest in the proof of Cheeger-Gromoll. When γ is a line, we will find that the Busemann function is harmonic, inducing rigidity on the geometry of its gradient. In this subsection, we prove basic properties of Busemann functions.

Lemma 3.4. The function $f(x) = d(x, p)$ is 1-Lipschitz, hence continuous.

Proof. Take arbitrary $x, y \in M$ and fix $p \in M$. The triangle inequality gives $d(x, p) \leq d(x, y) + d(y, p)$, hence $d(x, p) - d(y, p) \leq d(x, y)$. If we swap x and y, we also have that $d(y, p) - d(x, p) \leq d(y, x) = d(x, y)$, and therefore $|d(x, p) - d(y, p)| \leq d(x, y)$. If we let $f(x) = d(x, p)$, then we see that f is 1-Lipschitz, hence continuous: $|d(x, \gamma(t)) - d(y, \gamma(t))| = |f(x) - f(y)| \leq |x - y|$. □

Definition 3.5. For each ray $\gamma : [0, \infty) \to M$, define the function $g_t(x) = d(x, \gamma(t)) - t$ for $t \geq 0$.

Proposition 3.6. The function $g_t(x)$ is decreasing and bounded below with respect to t.

Proof. Suppose that $t < s$, and let γ be the ray associated with g_t. Then the triangle inequality gives that

$$d(x, \gamma(s)) \leq d(x, \gamma(t)) + d(\gamma(t), \gamma(s)) = d(x, \gamma(t)) + (s - t),$$

since γ is minimizing and unit speed. Rearranging, we have

$$g_s(x) = d(x, \gamma(s)) - s \leq d(x, \gamma(t)) - t = g_t(x).$$

Therefore for fixed x, the function $g_t(x)$ is decreasing with respect to t.

Next, we claim g_t is bounded below by $-d(x, \gamma(0))$ with respect to t. Fix $x \in M$, and note that the triangle inequality gives that $d(\gamma(0), \gamma(t)) - d(x, \gamma(0)) \leq d(x, \gamma(t))$ after rearrangement. Since γ is a unit speed ray, the segment $\gamma|_{[0,t]}$ is minimizing, hence $d(\gamma(0), \gamma(t)) = t$. Plugging in to the earlier inequality and rearranging gives that $d(x, \gamma(t)) - t \geq -d(x, \gamma(0))$; the left hand side is precisely $g_t(x)$. □

Proposition 3.7. The family $\{g_t\}$ is uniformly equicontinuous.

Proof. If we set $p = \gamma(t)$ and apply Lemma 3.4, we get that g_t is 1-Lipschitz, hence continuous. Fixing $\varepsilon > 0$, take $\delta = \varepsilon$. Then for all $t \geq 0$ and points x, y such that $d(x, y) < \delta$, we have that by the Lipschitz condition $|g_t(x) - g_t(y)| < \delta = \varepsilon$. Hence $\{g_t\}$ is uniformly equicontinuous. □

Definition 3.8. We define the *Busemann function of* γ to be $g_\gamma = \lim_{t\to\infty} g_t$.

Proposition 3.9. The functions g_t converge uniformly on compact sets to g_γ as $t \to \infty$.

Proof. Let $K \subset M$ be compact, and consider the restrictions $g_t : K \to \mathbb{R}$. Since $\{g_t\}$ is uniformly equicontinuous by Proposition 3.7, it is both equicontinuous and uniformly bounded. An application of Arzelà-Ascoli gives that for any sequence $t_n \to \infty$, there exists a subsequence t_{n_k} and a continuous function $h : K \to \mathbb{R}$ such that $g_{t_{n_k}} \to h$ uniformly on K.

Recall that g_t is bounded below via Proposition 3.6; we can make this uniform by taking a supremum:

$$C := -\sup_{x\in K} d(x, \gamma(0)) \leq -d(x, \gamma(0) \leq g_t(x).$$

Therefore for each x, the limit $g_\gamma(x) = \lim_{t\to\infty} g_t(x) = \inf_{t\geq 0} g_t(x)$ is finite, thus exists, by the monotone convergence theorem for sequences. If we take the subsequence $g_{t_{n_k}}$ that converges uniformly to h on K, we have that for each $x \in K$, $h(x) = \lim_{k\to\infty} g_{t_{n_k}}(x)$ since uniform convergence implies pointwise convergence. But $\{g_t(x)\}$ is monotonically decreasing with respect to t (see Proposition 3.6) and has limit $g_\gamma(x)$, and so every sequence $t_{n_k} \to \infty$ satisfies $g_\gamma(x) = \lim_{k\to\infty} g_{t_{n_k}}(x)$. Therefore $h = g_\gamma$ pointwise on K.

Finally, we want to show that the family $\{g_t\}$ converges uniformly on K. However this is immediate via Dini's theorem, which states that if K is a compact set and $g_t, g_\gamma \in C(K)$ are such that $g_t \to g_\gamma$ monotonically, then $g_t \to g_\gamma$ uniformly. □

Proposition 3.10. Busemann functions are 1-Lipschitz.

Proof. We have already shown that g_t is 1-Lipschitz via a simple modification of Lemma 3.4. Now for any $x, y \in M$, we have

$$|g_\gamma(x) - g_\gamma(y)| = \left|\lim_{t\to\infty} g_t(x) - \lim_{t\to\infty} g_t(y)\right| = \left|\lim_{t\to\infty}(g_t(x) - g_t(y))\right|.$$

Since $|\lim \alpha| \le \limsup |\alpha|$,

$$|g_\gamma(x) - g_\gamma(y)| \le \limsup_{t\to\infty} |g_t(x) - g_t(y)| \le \limsup_{t\to\infty} d(x,y) = d(x,y).$$

Therefore the Busemann function g_γ is 1-Lipschitz. □

Example 3.11. Consider the hyperbolic plane $\mathbb{H}$ and consider the ray $\gamma(t) = (0, e^t)$ starting at the point $(x, y) = (0, 1)$ and going straight up; since this is parameterized by arc length as in Example 1.1, $d((0,1),(0,e^t)) = t$ in the hyperbolic metric. Recall then that in this case, $g_\gamma(x,y) = \lim_{t\to\infty}(d((x,y),(x,e^t)) - t)$. To compute the distance between (x,y) and (x,e^t), note that $ds = dy/y$ (since $y > 0$), we have (after setting $x = 0$ without loss of generality)

$$d((0,y),(0,e^t)) = \int_y^{e^t} \frac{dy}{y} = \ln(e^t) - \ln(y) = t - \ln(y).$$

Therefore, $g_\gamma(x,y) = \lim_{t\to\infty}(t - \ln(y)) - t = -\ln y$.

Lemma 3.12. Fix $p \in M$, let $\sigma : [0,\ell] \to M$ be a minimizing geodesic from some x to p with initial velocity $w = \sigma'(0)$, and fix $u \in T_pM$. Then

$$\limsup_{s\searrow 0} \frac{d(\exp_x(su), p) - d(x,p)}{s} \le -\langle u, w\rangle.$$

Proof. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ defined by $\alpha(s) = \exp_x(su)$ be a variational curve where $u \in T_xM$, and take a smooth vector field J along σ such that $J(0) = u$ and $J(\ell) = 0$. Using J, we can define the variation $\Gamma(s,t) = \exp_{\sigma(t)}(sJ(t))$. Then $\Gamma(0,t) = \sigma(t)$, $\Gamma(s,0) = \exp_x(su) = \alpha(s)$, and $\Gamma(s,\ell) = \exp_p(0) = p$. So for each fixed s, the curve $\Gamma_s(t)$ connects $\alpha(s)$ to p, hence $d(\alpha(s),p) \le L_g(\Gamma_s)$. Since $L_g(\Gamma_0) = L_g(\sigma) = d(x,p)$, we have

$$\frac{d(\alpha(s),p) - d(x,p)}{s} \le \frac{L_g(\Gamma_s) - L_g(\Gamma_0)}{s}.$$

Taking $\limsup$ gives

$$\begin{aligned}\limsup_{s\searrow 0} \frac{d(\alpha(s),p) - d(x,p)}{s} &\le \left.\frac{d}{ds}\right|_{s=0} L_g(\Gamma_s) \\ &= \left\langle \frac{\partial\Gamma}{\partial s}(0,\ell), \sigma'(\ell)\right\rangle - \left\langle \frac{\partial\Gamma}{\partial s}(0,0), \sigma'(0)\right\rangle \\ &= -\langle u, w\rangle,\end{aligned}$$

where the first equality comes from the first variation formula, and the second equality is because $J(\ell) = 0$ and $J(0) = u$. □

Proposition 3.13. If g_γ is a Busemann function, then $|\nabla g_\gamma| = 1$ almost everywhere.

Proof. By Proposition 3.10, g_γ is 1-Lipschitz. Rademacher's theorem states that a Lipschitz function is differentiable almost everywhere, and the differential is bounded by the Lipschitz constant. In particular, we have that $\|d(g_\gamma)_x\| \leq 1$, where $\|\cdot\|$ is the operator norm. By definition of the gradient, $dg_\gamma(v) = \langle \nabla g_\gamma, v\rangle$, hence

$$\|d(g_\gamma)_x\| = \sup_{|v|=1} |d(g_\gamma)_x(v)| = \sup_{|v|=1} |\langle \nabla g_\gamma(x), v\rangle| = |\nabla g_\gamma(x)|.$$

Therefore $|\nabla g_\gamma| \leq 1$ a.e.

For the other inequality, fix a point $x \in M$ at which g_γ is differentiable. For each $t > 0$, take a minimizing unit-speed geodesic $\sigma_t : [0, \ell_t] \to M$ from x to $\gamma(t)$ such that $\sigma_t(0) = x$, $\sigma_t(\ell_t) = \gamma(t)$, and $\ell_t = d(x, \gamma(t))$. Let $v_t := \sigma_t'(0) \in T_pM$; this is a unit vector. Since $S_xM \subset T_xM$ is compact, there exists a subsequence $\{v_{t_j}\}$ of $\{v_t\}$ such that $v_{t_j} \to v \in T_xM$.

For each j, let $p_j := \gamma(t_j)$ be the target point, let $\sigma_{t_j} : [0, \ell_{t_j}] \to M$ be a minimizing geodesic from x to p_j, let $v_{t_j} := \sigma_{t_j}'(0)$ be the initial direction, and let $u := v$. Applying Lemma 3.12 yields

$$\limsup_{s \searrow 0} \frac{d(\exp_x(sv), \gamma(t_j)) - d(x, \gamma(t_j))}{s} \leq -\langle v, v_{t_j}\rangle.$$

Since $g_{t_j}(y) = d(y, \gamma(t_j)) - t_j$, subtracting the constant t_j does not effect the difference in the quotient, and so

$$\limsup_{s \searrow 0} \frac{g_{t_j}(\exp_x(sv)) - g_{t_j}(x)}{s} \leq -\langle v, v_{t_j}\rangle.$$

We now want to pass to the limit as $j \to \infty$. Fix $s > 0$; since $g_{t_j} \to g_\gamma$ pointwise (uniformly, in fact),

$$\frac{g_{t_j}(\exp_x(sv)) - g_{t_j}(x)}{s} \to \frac{g_\gamma(\exp_x(sv)) - g_\gamma(x)}{s}.$$

Also, since $v_{t_j} \to v$, we have $\langle v, v_{t_j}\rangle \to \langle v, v\rangle = 1$. Hence

$$\limsup_{s \searrow 0} \frac{g_\gamma(\exp_x(sv)) - g_\gamma(x)}{s} \leq -1.$$

Since g_γ is differentiable at x, the left hand side is equal to the directional derivative $D_v g_\gamma(x) = \langle \nabla g_\gamma(x), v\rangle$, hence $\langle \nabla g_\gamma(x), v\rangle \leq -1$. Applying Cauchy-Schwarz gives

$$1 \leq |\langle \nabla g_\gamma(x), v\rangle| \leq |\nabla g_\gamma(x)||v| = |\nabla g_\gamma(x)|$$

as desired. □

3.3. **Harmonicity and Smoothness of Busemann Functions for Lines.** Typically, Busemann functions are defined just on rays; if we extend the domain of a globally minimizing geodesic γ to be all of $(-\infty, \infty)$, that is $\gamma : \mathbb{R} \to M$ is a line, we get the powerful additional property of harmonicity on the two Busemann functions associated with the forward and backward rays when we restrict the domain of the line γ.

The strategy that we will follow is to employ the estimate of Theorem 2.18 to Δg_t. We would like to then take a limit in order to study the Laplacian of the Busemann function g_γ, with the goal of showing that g_γ is superharmonic (i.e., $\Delta g_\gamma \leq 0$) – and after a symmetry argument, harmonic. However, we do not necessarily have $\lim_{t\to\infty} \Delta g_t = \Delta g_\gamma$. For example,

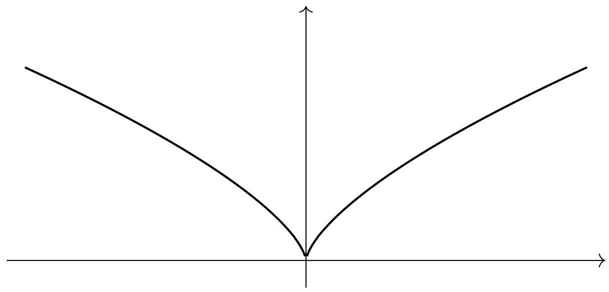

FIGURE 3. The graph of the function $f(x) = x^{2/3}$. Notice that $f''(x) < 0$ a.e., however f is not superharmonic around 0, since f contains its minimum $f(0)$ on the interior of any interval $[-a, a]$, contradicting the maximum (in fact, minimum) principle.

consider the function $f(x) = x^{2/3}$ as depicted in Figure 3: $f'' < 0$ almost everywhere, but f is not superharmonic, since the second derivative gives an infinite integral in a neighborhood of 0. Instead, we only get that g_γ is harmonic in a distributional sense, that is, when integrated against a compactly supported smooth test function.

This subsection is dedicated exclusively to applying the comparison estimate to Δg_t, and then rectifying the issue that a priori we cannot push the limit through the Laplacian. To do so, we will establish a seminal result for the regularity of solutions to elliptic partial differential equations in Euclidean space, and then extend this to Riemannian manifolds. This will allow us to upgrade from weak harmonicity on M to strong harmonicity on M.

Definition 3.14. Let $\gamma : \mathbb{R} \to M$ be a unit speed line. Define the *forward* Busemann function $g_+(t) = g_\gamma(t)$ with $t \in [0, \infty)$ and the *backward* Busemann function $g_-(t) = g_\gamma(-t)$ for $t \in [0, \infty)$.

Theorem 3.15. The Busemann functions g_+ and g_- are weakly harmonic when $\operatorname{Ric} \geq 0$.

Proof. The main piece of machinery we use is the Laplacian comparison of Theorem 2.18. Suppose that $\gamma : \mathbb{R} \to M$ is a unit speed line. Then for $\gamma|_{[0,\infty)}$, we define the radial distance function $r_t(x) = d(x, \gamma(t))$. If $g_t(x) = r_t(x) - t = d(x, \gamma(t)) - t$, then certainly $\Delta r_t = \Delta g_t$. Theorem 2.18 gives that away from the cut locus of $\gamma(t)$ and for each t,

$$\Delta g_t(x) = \Delta r_t(x) \leq \frac{n-1}{r_t(x)}.$$

Notice now that if we were to take a limit, we get

$$\lim_{t\to\infty} \Delta g_t(x) \leq \lim_{t\to\infty} \frac{n-1}{r_t(x)} = 0.$$

But, we do not have that $\lim_{t\to\infty} \Delta g_t = \Delta g_\gamma$, which would allow us to say that g_γ is strongly superharmonic (again reference Figure 3); we instead must try for weak superharmonicity.

Recall that for a locally integrable function f, the weak Laplacian of f is defined as $\langle \Delta f, \varphi \rangle := \langle f, \Delta\varphi \rangle = \int_M f \Delta\varphi \, d\mu$ for every nonnegative test function $\varphi \in C_c^\infty(M)$. Then for fixed $\varphi \geq 0$ where $\varphi \in C_c^\infty(M)$ with $\operatorname{supp}\varphi$ contained in some compact set K, we can take the comparison inequality and write it distributionally as

$$\langle \Delta g_t, \varphi \rangle = \langle g_t, \Delta\varphi \rangle \leq \left\langle \frac{n-1}{r_t}, \varphi \right\rangle,$$

or equivalently

$$\int_M g_t \Delta\varphi \leq \int_M \frac{n-1}{r_t(x)}\varphi.$$

We want to take a limit over t of this inequality and achieve $\int_M g_\gamma \Delta\varphi \leq 0$ for all nonnegative $\varphi \in C_c^\infty(M)$. For the left hand side, notice that

$$\left| \int_M g_t \Delta\varphi - \int_M g_\gamma \Delta\varphi \right| = \left| \int_K (g_t - g_\gamma)\Delta\varphi \right| \leq \|g_t - g_\gamma\|_{L^\infty(K)} \int_K |\Delta\varphi|.$$

By Proposition 3.9, $g_t \to g_\gamma$ uniformly on K, and so $\|g_t - g_\gamma\|_{L^\infty(K)} \to 0$. Also the integral $\int_K |\Delta\varphi|$ is finite since we are integrating a continuous function ($\Delta\varphi \in C^\infty(M)$ since $\varphi \in C^\infty(M)$) over a compact set, and therefore $g_t \to g_\gamma$ in $L^1_{\text{loc}}(M)$; that is, $\lim \int_M g_t \Delta\varphi = \int g_\gamma \Delta\varphi$. For the right hand side, fix $x \in K$. The reverse triangle inequality gives

$$r_t(x) = d(x, \gamma(t)) \geq t - d(x, \gamma(0)),$$

and so $r_t(x) \to \infty$ uniformly on K, hence $(n-1)/r_t(x) \to 0$ uniformly on K. So as $t \to \infty$,

$$\int_M g_\gamma \Delta\varphi = \langle \Delta g_\gamma, \varphi \rangle \leq 0$$

for all nonnegative $\varphi \in C_c^\infty(M)$, that is, g_γ is weakly superharmonic.

Consider the Busemann functions g_+ and g_- associated to the forward and backward rays of the line γ as defined in Definition 3.14. By the preceding argument, both g_+ and g_- are weakly superharmonic. Also, we have that

$$g_+(\gamma(s)) = \lim_{t\to\infty} (d(\gamma(s), \gamma(t)) - t) = \lim_{t\to\infty} (|t-s| - t) = -s,$$

since $|t-s| = t-s$ when $t > s$, and

$$g_-(\gamma(s)) = \lim_{t\to\infty} (d(\gamma(s), \gamma(-t)) - t) = \lim_{t\to\infty} (|s+t| - t) = s.$$

Therefore $g_+ + g_- = 0$ along the line γ, that is, $g_- = -g_+$ along γ. Since g_- is weakly superharmonic, so is $-g_+$. But that means both $\Delta g_+ \leq 0$ and $\Delta(-g_+) \leq 0$ distributionally, hence $\Delta g_+ = 0$ distributionally; that is, g_+ is weakly harmonic. Similarly, g_- is weakly harmonic. □

We would like to strengthen this result so that g_+ and g_- are strongly harmonic on the manifold M. In order to do so, we want to apply a regularity result of the (elliptic) Laplace-Beltrami operator. In particular, we have the following result that we can apply to the Laplacian on $\mathbb{R}^n$, which we generalize to Riemannian manifolds in the subsequent corollary, that we will use to assemble a desired result in the proof of Cheeger-Gromoll in Section 6. We give a sketch of the proof; for a full treatment, reference §6.3 of [Eva10].

Theorem 3.16 (Elliptic Regularity)**.** Let $Lu = -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu$ be a uniformly elliptic operator with smooth coefficients $a^{ij}, b^i, c \in C^\infty(U)$ for $i, j = 1, ..., n$, and $f \in C^\infty(U)$ where $U \subset \mathbb{R}^n$. Suppose $u \in H^1(U) = \{u \in L^2(U) : \partial_1 u, ..., \partial_n u \in L^2(U)\}$ is a weak solution of the elliptic PDE $Lu = f$ in U. Then $u \in C^\infty(U)$.

Proof Sketch. We proceed in three main stages. Throughout, L is assumed to be uniformly elliptic with constant $\theta > 0$. That is, $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$ for all $x \in U$ and $\xi \in \mathbb{R}^n$.

Step 1: A priori, $u \in H^1(U)$ has only *one* weak derivative in L^2. We want to upgrade to having *two* derivatives in L^2. Since u is defined only as a weak solution, we cannot differentiate $Lu = f$, but we can use difference quotients as a proxy. For small $h \neq 0$ and a coordinate direction e_k, define the difference quotient

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}.$$

Notably, $u \in H^1$ has a weak derivative $u_{x_k} \in L^2$ if and only if $\|D_k^h u\|_2$ is bounded uniformly in h (cf. Theorem 3(ii) of §5.8.2 [Eva10]). We want to establish such a bound on $D_k^h(Du)$.

Let $V \Subset W \Subset U$ and consider the smooth cutoff function $\zeta = 1$ on V and $\zeta = 0$ on $\mathbb{R}^n \setminus W$. Set $\tilde{f} := f - \sum_{i=1}^n b^i u_{x_i} - cu \in L^2(U)$, so that the weak formulation reduces to

$$\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j}\, dx = \int_U \tilde{f} v\, dx$$

for all $v \in H_c^1(U)$. Substituting the test function $v = -D_k^{-h}(\zeta^2 D_k^h u) \in H_c^1(U)$, we obtain the following equation:

$$A := \sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j}\, dx = \int_U \tilde{f} v\, dx =: B.$$

We can expand A using the product rule and the identity $\int v D_k^{-h} w = -\int w D_k^h v$ to isolate the leading term of A,

$$A_1 := \sum_{i,j=1}^n \int_U a^{ij}(x + he_k) D_k^h u_{x_i} D_k^h u_{x_j} \zeta^1\, dx,$$

and by uniform ellipticity, we bound

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2\, dx.$$

The remaining terms of A and the right hand side B are estimated by Cauchy's inequality with ε taken to be a fraction of θ in order to be absorbed in the the left hand side. In particular, we have

$$\int_V |D_k^h Du|^2\, dx \leq C \int_U (f^2 + u^2 + |Du|^2)\, dx$$

uniformly in h. Taking $h \to 0$ gives $Du \in H^1_{\mathrm{loc}}(U)$, hence $u \in H^2_{\mathrm{loc}}(U)$.

Step 2: We want to iterate Step 1 to get an arbitrary Sobolev regularity result. The H^2_{loc} result of Step 1 is the base case $m = 0$. To reach H^{m+2}_{loc} for all m, we proceed by induction. Suppose $u \in H^{m+2}_{\text{loc}}(U)$. Then for any multi-index α such that $|\alpha| = m+1$, we can differentiate $Lu = f$ by D^α to get the PDE $L(D^\alpha u) = \tilde{f}$ in W, where

$$\tilde{f} := D^\alpha f + \sum_{\beta<\alpha} \binom{\alpha}{\beta} (\text{commutator terms})$$

involves derivatives of the coefficients a^{ij}, b^i, c applied to lower order derivatives of u. Since we assume $a^{ij}, b^i, c \in C^\infty(U)$, these commutator terms are controlled, and the inductive hypothesis ensures that all lower order derivative terms of u appearing in $\tilde{f}$ lie in L^2_{loc}. So $\tilde{u} := D^\alpha u \in H^1_{\text{loc}}(W)$ is a weak solution of $L\tilde{u} = \tilde{f}$ with $\tilde{f} \in L^2_{\text{loc}}$. Applying Step 1 gives that $u \in H^{m+3}_{\text{loc}}$. Since $f \in C^\infty(U) \subset H^m(U)$, and since $a^{ij}, b^i, c \in C^\infty(U)$ for all m, we conclude $u \in H^k_{\text{loc}}(U)$ for all $k \geq 1$.

Step 3: Although we have membership in all Sobolev spaces, we do not immediately have that u is smooth – to achieve this, we use Sobolev embedding techniques. Fix $\ell \in \mathbb{N}$ and take an integer $k > n/2$ such that $k - \lfloor n/2 \rfloor - 1 \geq \ell$. Step 2 gives $u \in H^k_{\text{loc}}(U) = W^{k,2}_{\text{loc}}(U)$. We iteratively apply the Gagliardo-Nirenberg-Sobolev inequality (Theorem 1 in §5.6.1 [Eva10]) starting with $p_0 = 2$; at each step, the inequality gives $D^\beta u \in L^{p*}$ where $1/p^* = 1/p - 1/n$ and $|\beta| = k - 1$, so that $u \in W^{k-1,p*}_{\text{loc}}$.

Setting $p_1 = p^*$, $p_2 = p_1^*$, and so on, after $\lfloor n/2 \rfloor$ steps the exponent satisfies $p_{\lfloor n/2 \rfloor} > n$, since each application of $p \mapsto p^*$ increases $1/p$ by $1/n$, and we have subtracted $\lfloor n/2 \rfloor / n$ from $1/2$ in total, hence we reach 0. That is, we have gotten to $W^{k-\lfloor n/2 \rfloor, r}_{\text{loc}}$ with $r > n$, and Morrey's inequality (Theorem 4 in §5.6.2 [Eva10]) gives the embedding into $C^{k-\lfloor n/2 \rfloor -1,\delta}(U)$ for some $\delta \in (0, 1]$. In particular, $u \in C^\ell_{\text{loc}}(U) = C^\ell(U)$ since classical differentiability is a local condition. Since $\ell > 0$ was arbitrary, $u \in C^\infty(U)$. □

We now want to generalize this to the case of Riemannian manifolds in order to apply the strengthening result to the weakly harmonic functions $g_\pm$. First, however, we need to show that the pushforward of the Busemann function g_+ to Euclidean space belongs to the Sobolev space $H^1(U)$.

Lemma 3.17. Let (M, g) be a Riemannian manifold and g_+ the Busemann function associated to a ray γ. Let $\varphi : U \to V \subset \mathbb{R}^n$ be a coordinate chart, and suppose V is convex. Then $\tilde{u} := g_+ \circ \varphi^{-1} \in H^1_{\text{loc}}(V)$ – that is, $\tilde{u} \in H^1(W)$ for all $W \Subset V$.

Proof. We will show $\tilde{u} \in H^1(W)$ for all $W \Subset V$. Fix such a $W \Subset V$; we can then find an intermediate V' such that $W \Subset V' \Subset V$ such that $\overline{V'}$ is compact and convex. Set $K = \varphi^{-1}(\overline{V'}) \Subset U$. K is compact since φ is a diffeomorphism. Since $g_{ij} \in C^\infty(U)$ and K is compact, the matrix $(g_{ij}(p))$ is uniformly bounded and uniformly positive definite over $p \in K$. That is, there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda |\xi|^2 \leq g_{ij}(p) \xi^i \xi^j \leq \Lambda |\xi|^2$$

for all $p \in K$ and $\xi = (\xi^1, ..., \xi^n) \in \mathbb{R}^n$. For $x, y \in \overline{W}$, the convexity of $\overline{V'}$ gives $\sigma(t) := (1-t)x + ty \subset \overline{V'}$ with $t \in [0,1]$ and its preimage $c(t) := \varphi^{-1}(\sigma(t))$ lies in K. We can use this curve c to derive the following upper bound:

$$\begin{aligned} d_g(\varphi^{-1}(x), \varphi^{-1}(y)) &\leq \int_0^1 \sqrt{g_{ij}(\varphi^{-1}(\sigma(t)))\dot{c}^i\dot{c}^j}\,dt && d_g \leq L_g \\ &\leq \int_0^1 \sqrt{g_{ij}(\varphi^{-1}(\sigma(t)))(y^i - x^i)(y^j - x^j)}\,dt && d\varphi_{c(t)}(\dot{c}(t)) = y - x \\ &\leq \int_0^1 \sqrt{\Lambda}|x-y|_{\mathbb{R}^n}\,dt && g_{ij}(p)\xi^i\xi^j \leq \Lambda|\xi|^2 \\ &= \sqrt{\Lambda}|x-y|_{\mathbb{R}^n}. \end{aligned}$$

Since g_+ is 1-Lipschitz (Proposition 3.10), we get

$$|\tilde{u}(x) - \tilde{u}(y)| = |g_+(\varphi^{-1}(x)) - g_+(\varphi^{-1}(y))| \leq d_g(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \sqrt{\Lambda}|x-y|$$

for $x, y \in \overline{W}$, and so $\tilde{u}$ is $\sqrt{\Lambda}$-Lipschitz on $\overline{W}$. Thus $\tilde{u} \in W^{1,\infty}(W) \subset W^{1,2}(W) = H^1(W)$. □

We can now extend the regularity result of Theorem 3.16 to the Laplace-Beltrami operator on a Riemannian manifold. We denote this by Δ in the following corollary.

Corollary 3.18 (Weyl's Lemma for g_+ on a Riemannian Manifold)**.** Let (M, g) be a Riemannian manifold and $g_+ \in D'(M)$ is the forward Busemann function associated to a ray γ. If $\langle g_+, \Delta\varphi \rangle = 0$ for all nonnegative $\varphi \in C_c^\infty(M)$, then $g_+ \in C^\infty(M)$ and $\Delta g_+ = 0$.

Proof. It suffices to show that g_+ is smooth near each $p \in M$. Let (U, ψ) be a chart around p; let $\psi(U) = \Omega \subset \mathbb{R}^n$ be convex (e.g., Ω is an open ball), and set $\tilde{u} = g_+ \circ \psi^{-1} \in D'(\Omega)$. In the chart U, we can write the Laplace-Beltrami operator in divergence form as

$$\Delta g_+ = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ij}\partial_j g_+\right).$$

If we define $\Delta g_+ = L\tilde{u}$, we can expand to get the following expression for L in coordinates:

$$L = \frac{1}{\sqrt{\det g}}\left(\partial_i\left(\sqrt{g}g^{ij}\right)\partial_j g_+ + \sqrt{\det g}g^{ij}\partial_i\partial_j g_+\right) = g^{ij}\partial_{ij} + \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ij}\right)\partial_j.$$

If we let $a^{ij} = g^{ij}$ and $b^i = \frac{1}{\sqrt{\det g}}\partial_j(\sqrt{\det g}g^{ij})$, then

$$L = a^{ij}(x)\partial_{ij} + b^i(x)\partial_i.$$

Notice that both $a^{ij}, b^i \in C^\infty(\Omega)$ since g is smooth.

Now fix $\Omega'' \Subset \Omega' \Subset \Omega$, and define $K = \psi^{-1}(\overline{\Omega'})$, which is compact in M since ψ is a diffeomorphism. Since K is compact and g_{ij} is smooth, there exist $0 < \lambda \leq \Lambda < \infty$ such that $\lambda|\xi|^2 \leq g_{ij}(q)\xi^i\xi^j \leq \Lambda|\xi|^2$ for all $q \in K$ and $\xi \in \mathbb{R}^n$. We can make the substitution $q = \psi^{-1}(x)$ and invert this inequality to get

$$\frac{1}{\Lambda}|\xi|^2 \leq g^{ij}(x)\xi_i\xi_j = a^{ij}\xi_i\xi_j \leq \frac{1}{\lambda}|\xi|^2$$

for all $x \in \overline{\Omega'}$ (since ψ^{-1} is a bijection between $\overline{\Omega'}$ and K), which gives that L is a uniformly elliptic operator on Ω'.

We have already shown in Lemma 3.17 that $\tilde{u} \in H^1_{\text{loc}}(\Omega)$, in particular $\tilde{u} \in H^1(\Omega')$. It remains to show that $\tilde{u}$ is a weak solution to $L\tilde{u} = 0$ on Ω' in order to apply Theorem 3.16.

For any $\varphi \in C_c^\infty(\Omega')$, let us set $\Phi = \varphi \circ \psi \in C_c^\infty(U) \subset C_c^\infty(M)$. Then since $\langle g_+, \Delta\Phi\rangle = 0$ by Theorem 3.15, by using the change of variables $q = \psi^{-1}(x)$ we can write

$$0 = \int_M g_+ \Delta\Phi \, dV_g = \int_{\Omega'} \tilde{u} \cdot (L\varphi)\sqrt{\det g}\, dx,$$

which is equivalent to $\tilde{u}$ being a weak solution to $L\tilde{u} = 0$ on Ω'.

Since L has been shown to be uniformly elliptic on Ω' with smooth coefficients, and $\tilde{u} \in H^1(\Omega')$ is a weak solution to $L\tilde{u} = 0$, Theorem 3.16 gives that $\tilde{u} \in C^\infty(\Omega'')$. Since $\Omega'' \Subset \Omega' \Subset \Omega$ were arbitrary, $\tilde{u} \in C^\infty(\Omega)$, hence $g_+ \in C^\infty(U)$ since $g_+ = \tilde{u} \circ \psi$ and ψ is a diffeomorphism. Since $p \in M$ was arbitrary, $g_+ \in C^\infty(M)$. Finally, Green's second identity gives that $\Delta u = 0$ classically since for all test functions $\varphi \in C_c^\infty(M)$,

$$0 = \int_M g_+ \Delta\varphi = \int_M \varphi \Delta g_+.$$

□

4. Analytic Rigidity via Bochner's Formula

We have established that the Busemann function g_+ associated to a line γ is harmonic in the nonnegative Ricci curvature setting. However, we need a way to move from harmonicity to some form of splitting result. The first main step toward this goal is the Bochner formula. This formula will convert the harmonicity condition into geometric information about ∇g_+. In particular, we will see that Bochner's formula ensures that ∇g_+ is a parallel vector field, and then Section 5 will discuss how we turn this parallel vector field into a splitting result. We begin by stating and proving Bochner's formula.

Theorem 4.1 (Bochner's Formula)**.** Let u be a smooth function on the manifold M. Then

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess}(u)|^2 + \langle \nabla\Delta u, \nabla u\rangle + \operatorname{Ric}(\nabla u, \nabla u).$$

Proof. Fix $p \in M$, and let $\{E_1, ..., E_n\}$ be an orthonormal frame, that is $\langle E_i, E_j\rangle = \delta_i^j$ and $\nabla_{E_i} E_j(p) = 0$. Recall that for $Y \in \mathcal{X}(M)$, we define $\operatorname{div} Y = \sum \langle \nabla_{E_i} Y, E_i\rangle$, and also recall $\Delta f = \operatorname{div} \nabla f$. So for $Y = \nabla f$, we have

$$\Delta f = \operatorname{div}(\nabla f) = \sum_i \langle \nabla_{E_i} \nabla f, E_i\rangle.$$

By metric compatibility, we know $E_i\langle \nabla f, E_i\rangle = \langle \nabla_{E_i}\nabla f, E_i\rangle + \langle \nabla f, \nabla_{E_i} E_i\rangle$, hence

$$\langle \nabla_{E_i}\nabla f, E_i\rangle = E_i\langle \nabla f, E_i\rangle - \langle \nabla f, \nabla_{E_i} E_i\rangle.$$

Since $E_i f = \langle \nabla f, E_i\rangle$, the first term on the right becomes $E_i(E_i f)$, and therefore

$$\Delta f = \sum_i E_i E_i f - \langle \nabla_{E_i} E_i, \nabla f\rangle.$$

Letting $f = |\nabla u|^2$, we have

$$\Delta |\nabla u|^2 = \sum E_i E_i \langle \nabla u, \nabla u \rangle - \sum \langle \nabla_{E_i} E_i, \nabla |\nabla u|^2 \rangle.$$

But, since we are working in an orthonormal frame about p, the second term is 0.

Proceeding with computation at the point p,

$$\begin{aligned}
\frac{1}{2}\Delta |\nabla u|^2 &= \frac{1}{2}\sum_i E_i E_i \langle \nabla u, \nabla u \rangle \\
&= \sum_i \langle \nabla_{E_i} \nabla u, \nabla u \rangle && \text{product rule} \\
&= \sum_i E_i \operatorname{Hess}(u)(E_i, \nabla u) && \text{def. of Hess} \\
&= \sum_i E_i \operatorname{Hess}(u)(\nabla u, E_i) && \text{Hess is symmetric} \\
&= \sum_i E_i \langle \nabla_{\nabla u}(\nabla u), E_i \rangle && \text{def. of Hess} \\
&= \sum_i \langle \nabla_{E_i} \nabla_{\nabla u}(\nabla u), E_i \rangle + \langle \nabla_{\nabla u}(\nabla u), \nabla_{E_i} E_i \rangle && \text{product rule} \\
&= \sum_i \langle \nabla_{E_i} \nabla_{\nabla u}(\nabla u), E_i \rangle && \text{orthonormality at } p \\
&= \sum_i \langle R(E_i, \nabla u)\nabla u, E_i \rangle \\
&\qquad + \sum_i \langle \nabla_{\nabla u} \nabla_{E_i}(\nabla u), E_i \rangle \\
&\qquad\qquad + \sum_i \langle \nabla_{[E_i, \nabla u]}(\nabla u), E_i \rangle,
\end{aligned}$$

where the last equality comes from the definition of the Riemann curvature tensor. The first term is by definition $\operatorname{Ric}(\nabla u, \nabla u)$. By the product rule, the second term is

$$\begin{aligned}
\sum_i \langle \nabla_{\nabla u} \nabla_{E_i}(\nabla u), E_i \rangle &= \sum_i (\nabla u) \langle \nabla_{E_i}(\nabla u), E_i \rangle - \langle \nabla_{E_i}(\nabla u), \nabla_{\nabla u} E_i \rangle \\
&= (\nabla u) \sum_i \langle \nabla_{E_i} \nabla u, E_i \rangle \\
&= (\nabla u) \Delta u \\
&= \langle \nabla \Delta u, \nabla u \rangle,
\end{aligned}$$

where the second equality comes from the fact that the E_i are parallel, hence $\nabla_X E_i = 0$ for any $X \in \mathcal{X}(M)$, the third equality comes from the definition we wrote in the first display-style line of the proof, and the last comes by definition and symmetry of the inner product.

Finally for the third term, we have

$$\sum_i \langle \nabla_{[E_i, \nabla u]}(\nabla u), E_i \rangle = \sum_i \operatorname{Hess}(u)([E_i, \nabla u], E_i) \qquad \text{def. of Hess}$$

$$
\begin{aligned}
&= \sum_i \operatorname{Hess}(u)(\nabla_{E_i}\nabla u - \nabla_{\nabla u}E_i, E_i) && \text{def. of Lie bracket}\\
&= \sum_i \operatorname{Hess}(u)(\nabla_{E_i}\nabla u, E_i) - \operatorname{Hess}(u)(\nabla_{\nabla u}E_i, E_i) && \text{linearity}\\
&= \sum_i \operatorname{Hess}(u)(\nabla_{E_i}\nabla u, E_i) && \text{parallel frame at } p\\
&= \sum_i \operatorname{Hess}(u)(E_i, \nabla_{E_i}\nabla u) && \text{symmetry}\\
&= \sum_i \langle \nabla_{E_i}\nabla u, \nabla_{E_i}\nabla u\rangle && \text{def. of Hess}\\
&= |\operatorname{Hess}(u)|^2.
\end{aligned}
$$

□

Notice that if we are working with a harmonic function u, then the term $\langle \nabla\Delta u, \nabla u\rangle = 0$. Therefore, we can reduce the Bochner formula in the case of harmonic u to

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess}(u)|^2 + \operatorname{Ric}(\nabla u, \nabla u).$$

Let us apply this result to the Busemann function g_+ on the manifold M, which we take to have nonnegative Ricci curvature. Recall that by Proposition 3.13, $|\nabla g_+| = 1$. Therefore on the left hand side of the reduced Bochner formula, we have $\frac{1}{2}\Delta(1) = 0$, and so

$$0 = |\operatorname{Hess}(g_+)|^2 + \operatorname{Ric}(\nabla g_+, \nabla g_+).$$

By definition, $|\operatorname{Hess}(u)|^2 = |\nabla(\nabla u)|^2 = \sum_i |\nabla_{E_i}\nabla u|^2$ for any orthonormal frame $\{E_i\}$. But since the left hand side is 0 and each of the summands on the right is nonnegative (surely $|\operatorname{Hess}(g_+)|^2 \geq 0$, and $\operatorname{Ric} \geq 0$ by assumption), we must have that $|\operatorname{Hess}(g_+)|^2 = \operatorname{Ric}(\nabla g_+, \nabla g_+) = 0$; in particular $\nabla_{E_i}\nabla g_+ \equiv 0$ for all i. Since the covariant derivative is linear in the lower slot, we have that $\nabla_X \nabla g_+ = \sum_i X^i \nabla_{E_i}\nabla g_+$ for any arbitrary $X \in \mathcal{X}(M)$, hence this vanishes as well. Since $\nabla_X\nabla g_+ = 0$ for all $X \in \mathcal{X}(M)$, ∇g_+ is parallel.

5. Geometric Rigidity via de Rham Decomposition

To show that the existence of a parallel vector field induces a splitting, we appeal to a weakened version of the following theorem (it is stated in [Bes87] that the proof of this theorem is very hard, and that a simple proof does not exist – it appears that this comment sparked further study, and a proof is given in [Pan92]):

Theorem 5.1 (de Rham)**.** If a Riemannian manifold is complete, simply connected, and if its holonomy representation is reducible, then (M, g) is a Riemannian product.

This theorem imposes the condition that M be simply connected; however, we can loosen the theorem by considering a local result without this hypothesis, and then instead by using properties of the Busemann function g_+, we can re-strengthen to a global result in this particular case. The local result follows:

Theorem 5.2 (Local de Rham)**.** Let (M, g) be a Riemannian manifold, not necessarily complete, and fix $p \in M$. Let $V \subset T_pM$ be the subspace of T_pM which is acted on trivially by $\mathrm{Hol}(p)$ (i.e., every vector in V should be fixed by action by any element of $\mathrm{Hol}(p)$), and let $V^\perp$ be its orthogonal complement in T_pM. Then M is locally a product with metric $\bar{g} \times g_\perp$.

In other words, if the holonomy representation $\mathrm{Hol}(p)$ is reducible, then the manifold locally splits as a Riemannian product. Before we prove this theorem, note that the machinery that we want to use in our argument is a generalization of vector fields – distributions (but not the kind used in Section 3). Consider the following definitions and fact about distributions.

Definition 5.3. If M is a smooth manifold, then a *(tangent) distribution* $\mathcal{D}$ assigns to each $p \in M$ a vector subspace $\mathcal{D}_p$ of T_pM in a smooth way. In particular, for any $p \in M$ there exists a neighborhood $U_p \subset M$ and a collection of vector fields $\{X_1, ..., X_k\}$ that span $\{X_1(q), ..., X_k(q)\} = \mathcal{D}_q$ for any $q \in U_p$.

The distribution is said to be *parallel* if it is parallel transport invariant; equivalently, if Y is a local section of $\mathcal{D}$ (i.e., $Y_p \in \mathcal{D}_p$ for all $p \in U \subset M$) and X is any vector field, then $\nabla_X Y$ is also a local section of $\mathcal{D}$.

The distribution $\mathcal{D}$ is said to be *involutive* if for any two $X, Y \in \Gamma(\mathcal{D}) \subset \mathcal{X}(M)$, their Lie bracket $[X, Y]$ is also contained in $\Gamma(\mathcal{D}) \subset \mathcal{X}(M)$.

If for every $\mathcal{D}_p$, we have that there exists some submanifold $N \subset M$ such that $\mathcal{D}_p = T_pN$, then we say that $\mathcal{D}$ is a *foliation*, and the submanifolds N are called *leaves* of the foliation.

Theorem 5.4 (Frobenius)**.** Every involutive distribution is (completely) integrable to a foliation.

Proof. We omit proof; see Theorem 19.12 in [Lee13]. □

In order to interface between the hypothesis about holonomy representation and distributions, we need the following lemma, which provides the link between ∇g_+ being parallel and the splitting result we desire.

Lemma 5.5. Let $k \in \mathbb{Z}$ be between 1 and $n-1$. Then the following are equivalent:

a) There exists on (M, g) a parallel, involutive distribution $\mathcal{D}$,
b) The holonomy representation $\mathrm{Hol}(g)$ leaves invariant a subspace of dimension k. Moreover, such a distribution is necessarily involutive.

Proof. (a $\Longrightarrow$ b) Fix $p \in M$ and consider $\mathcal{D}_p \subset T_pM$, a k-dimensional subspace of T_pM. If σ is a loop based at p, then we have an element $P_\sigma \in \mathrm{Hol}_p(g)$. Since $\mathcal{D}$ is parallel by assumption, parallel transport preserves $\mathcal{D}_p$ along every curve, that is, $P_\sigma(\mathcal{D}_p) = \mathcal{D}_p$. So every P_σ leaves $\mathcal{D}_p$ invariant, and so $\mathrm{Hol}_p(g)$ preserves $\mathcal{D}_p$. Since p is arbitrary, we have (b).

(b $\Longrightarrow$ a) Fix $p \in M$ and suppose $\mathrm{Hol}_p(g)$ preserves some k-dimensional $V \subset T_pM$. Define a distribution $\mathcal{D}$ on M by parallel transport: for any $q \in M$, pick some path σ from $p \to q$, and set $P_\sigma(\mathcal{D}_p) = \mathcal{D}_p$. So, every P_σ leaves $\mathcal{D}_p$ invariant by construction, and so $\mathrm{Hol}_p(g)$ preserves $\mathcal{D}_p$. Since p is arbitrary, we have (a).

To see that such a distribution $\mathcal{D}$ is involutive, consider two vector fields X, Y belonging to $\mathcal{D}$. Then both $\nabla_X Y$ and $\nabla_Y X$ belong to $\mathcal{D}$ by invariance under parallel transport, hence $[X, Y] = \nabla_X Y - \nabla_Y X$ belongs to $\mathcal{D}$. □

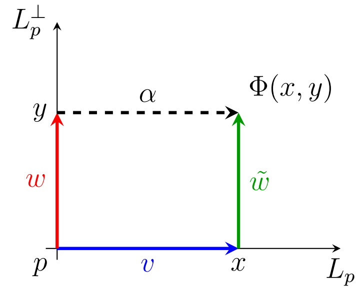

FIGURE 4. Construction of the map Φ

We can now proceed with the proof of the local de Rham splitting theorem.

Proof of Theorem 5.2. Fix $p \in M$ and assume that we can split $T_pM = V \oplus V^\perp$, where V is invariant under the action of $\mathrm{Hol}(p)$ (i.e., parallel transport around any loop based at p fixes V).

Our first goal is to build a totally geodesic, flat foliation. By Lemma 5.5, there exists a globally defined smooth involutive distribution $\mathcal{D} \subset TM$ such that $\mathcal{D}_p = V$ and $\nabla_X Y \in \Gamma(\mathcal{D})$ whenever $X, Y \in \Gamma(\mathcal{D})$ (parallel transport invariance). By the same logic, there also exists a globally defined smooth involutive distribution $\mathcal{D}^\perp \subset TM$ with the same properties. Since $\mathcal{D}, \mathcal{D}^\perp$ are involutive, Frobenius' Theorem (Theorem 5.4) gives that we can integrate these distributions to foliations: in particular, $\mathcal{D}$ integrates to a foliation $\mathcal{F}$, and $\mathcal{D}^\perp$ integrates to a foliation $\mathcal{F}^\perp$, and these two foliations are orthogonal.

Claim 1A: Every leaf of $\mathcal{F}$ is totally geodesic.

Proof. Let $L \subset \mathcal{F}$ be a leaf; by definition, $T_pL = \mathcal{D}_p$ for all $p \in L$, hence $TL = \mathcal{D}|_L$. So, if $X \in \mathcal{X}(L)$ is tangent to L, meaning $X_p \in T_pL$ for all p, then $X_p \in \mathcal{D}_p$. Recall Gauss' formula, $\nabla^M_X Y = \nabla^L_X Y + \mathrm{II}(X,Y)$. If $\nabla_X Y \in \Gamma(\mathcal{D})$ for $X, Y \in \Gamma(\mathcal{D})$, then X and Y are tangent to L since $\mathcal{D}|_L = TL$. Hence X and Y are $\mathcal{D}$-valued, so $\nabla^M_X Y \in \Gamma(\mathcal{D}) = \Gamma(TL)$, thus $\mathrm{II}(X,Y) = 0$. Equivalently, L is totally geodesic in M. $\diamond$

Claim 1B: The leaves of $\mathcal{F}$ are flat.

Proof. Since V is holonomy invariant, curvature must vanish in the directions of V; in particular, for any $v \in V$, we have $R(X,Y)v = 0$ for all $X, Y \in \mathcal{X}(M)$, hence all sectional curvatures of planes in $\mathcal{D}$ are 0. Since the leaves of a parallel distribution's foliation are totally geodesic by Claim 1A, the geodesics of L are geodesics of M. So, the curvature of the leaf is the same as the ambient curvature, that is $R^L(X,Y)Z = R(X,Y)Z$ with $X, Y, Z \in \mathcal{D}$. Hence $R^L(X,Y)v = 0$ for $v \in V$, and the leaves are flat. $\diamond$

Since the leaves are totally geodesic, $\exp^L_p(v) = \exp^M_p(v)$ for $v \in T_pL$, and so throughout when the domain of exp is ambiguous, we do not in fact run in to any issue. Further, the geodesic $\gamma_v(t) = \exp_p(tv)$ is contained entirely in L (respectively $L^\perp$ if $v \in L^\perp$), hence $\exp_p(T_pL) \subset L$, and $\exp_p(T_pL^\perp) \subset L^\perp$.

Our second goal is to build a local isometry. Consider the map $\Phi : L \times L^{\perp} \to M$, in which we somewhat combine the coordinates of a point $x \in L$ near p and a point $y \in L^{\perp}$ near p, by first moving along L and then parallel transporting along the $L^{\perp}$ direction. In particular, there exists a unique $v \in T_pL$ such that $y = \exp_p(v)$, and also there exists a unique $w \in T_pL^{\perp}$ such that $y = \exp_p(w)$. Suppose we parallel transport w along the geodesic $\alpha(t) = \exp_p(tv)$ (we can do this since the distributions are parallel), and call this $\tilde{w} \in T_xL_x^{\perp}$ – note here that we write $L_x^{\perp}$, which denotes the leaf of the foliation $\mathcal{F}^{\perp}$ passing through x, instead of the leaf passing through p, which retains the label $L^{\perp}$. Then we can finally define $\Phi(x, y) = \exp_x(\tilde{w})$, as illustrated in Figure 4.

We want to show that $\Phi^*g = g_L \oplus g_{L^{\perp}}$. Notice that $(\Phi^*g)_{(x,y)}(U, V) = g_{\Phi(x,y)}(d\Phi(U), d\Phi(V))$; so, we want to understand $d\Phi$. At a point $(x, y) \in L \times L^{\perp}$, we have the splitting $T_{(x,y)}(L \times L^{\perp}) = T_xL \oplus T_yL^{\perp}$. Suppose $(X_1, 0) \in T_xL$ and $(0, X_2) \in T_yL^{\perp}$. Recall that if $i : L \to M$ is an inclusion (immersion), then for any $x \in L$, we have $di_x(T_xL) = T_xL \subset T_xM$, and since $T_xL = \mathcal{D}_x$ by definition, $\dot{\alpha}(s) \in \mathcal{D}_{\alpha(s)}$ for any curve $\alpha(s) \subset L$. By construction of Φ, moving along the L direction pushes forward to a vector in $\mathcal{D}$ and moving along $L^{\perp}$ pushes forward to a vector in $\mathcal{D}^{\perp}$. In particular, $d\Phi(X_1, 0) \in \mathcal{D}_{\Phi(x,y)}$ and $d\Phi(0, X_2) \in \mathcal{D}^{\perp}_{\Phi(x,y)}$.

Consider the following expansion:

$$\begin{aligned}\langle d\Phi(X_1, X_2), d\Phi(Y_1, Y_2)\rangle_g &= \langle d\Phi(X_1, 0), d\Phi(Y_1, 0)\rangle_g + \langle d\Phi(X_1, 0), d\Phi(0, Y_2)\rangle_g \\ &\quad + \langle d\Phi(0, X_2), d\Phi(Y_1, 0)\rangle_g + \langle d\Phi(0, X_2), d\Phi(0, Y_2)\rangle_g \\ &= \langle d\Phi(X_1, 0), d\Phi(Y_1, 0)\rangle_g + \langle d\Phi(0, X_2), d\Phi(0, Y_2)\rangle_g,\end{aligned}$$

where the two cross terms cancel by orthogonality. We need to show that the right hand side is equivalent to $\bar{g}(X_1, Y_1) + g_{L^{\perp}}(X_2, Y_2)$. In particular, we want

$$\bar{g}(X_1, Y_1) = \langle d\Phi(X_1, 0), d\Phi(Y_1, 0)\rangle_g, \quad g_{L^{\perp}}(X_2, Y_2) = \langle d\Phi(0, X_2), d\Phi(0, Y_2)\rangle_g.$$

Notice that by fixing a variable of Φ, we can parameterize the leaves L and $L^{\perp}$. That is, we can locally define $\Phi : L \times L^{\perp} \to M$ so that for fixed $y \in L^{\perp}$, the map $\Phi_y(x) = \Phi(x, y)$ parameterizes L_y, and for fixed $x \in L$, the map $\Phi_x(y) = \Phi(x, y)$ parameterizes $L_x^{\perp}$.
Claim 2: Let $X_1, Y_1 \in T_xL$, and fix $y \in L^{\perp}$. Then the map $\Phi_y : L \to M$ preserves the metric on L, and the map $\Phi_x : L^{\perp} \to M$ preserves the metric on $L^{\perp}$.

Proof. Fix $y = \exp_p(w) \in L^{\perp}$ where $w \in T_pL^{\perp}$. For $x \in L$, take α_x to be the geodesic in L from p to x, and let $P_x : T_pM \to T_xM$ denote parallel transport along α_x. By definition, $\Phi_y(x) = \exp_x(P_xw)$. Since the splitting $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$ is parallel, P_x preserves $T_pL^{\perp}$, and so $P_xw \in T_xL^{\perp}$. Therefore $\Phi_y(x)$ is obtained by moving from x along the $L^{\perp}$ leaf through x.

Take a curve $x(s) \subset L$ such that $x(0) = x$ and $\dot{x}(0) = X_1 \in T_xL$, and fix $y \in L^{\perp}$. For each s, let α_s be the geodesic in L from p to $x(s)$, and let $\tilde{w}(s)$ denote the parallel transport of w along α_s. Then $\Phi(x(s), y) = \exp_{x(s)}(\tilde{w}(s))$. We can then define the variation $\Gamma(s, t) = \exp_{x(s)}(t\tilde{w}(s))$; by definition, each $\Gamma_s(t)$ is a geodesic, and $\Gamma(s, 1) = \Phi(x(s), y)$. Therefore,

$$d\Phi_{(x,y)}(X_1, 0) = \frac{d}{ds}\bigg|_0 \Phi(x(s), y) = \frac{\partial F}{\partial s}(0, t).$$

So, if we write the variation field $J_{X_1}(t) := \frac{\partial F}{\partial s}(0,t)$, then $d\Phi_{(x,y)}(X_1, 0) = J_{X_1}(1)$. Similarly for another vector $Y_1 \in T_xL$, we can construct and define a variation field $J_{Y_1}(t)$ such that $d\Phi_{(x,y)}(Y_1, 0) = J_{Y_1}(1)$.

We claim that $\langle J_{X_1}(1), J_{Y_1}(1)\rangle = \bar{g}(X_1, Y_1)$. At $s = 0$, we can define $\gamma(t) := \Gamma(0,t) = \exp_x(t\tilde{w})$; this is a the geodesic starting at x with initial velocity $\tilde{w}$. Since $L^\perp$ is totally geodesic, γ lies in the leaf $L_x^\perp$, and $\dot{\gamma}(t) \in \mathcal{D}^\perp$. Since the variation is obtained by moving the base point $x(s)$ inside L, the variation field $J_{X_1}(t)$ points in the $\mathcal{D}$ direction; that is $J_{X_1}(t), J_{Y_1}(t) \in \mathcal{D}_{\gamma(t)}$. Since $\mathcal{D}$ is a parallel distribution, parallel transport preserves $\mathcal{D}$. In particular along γ tangent to $\mathcal{D}^\perp$, the $\mathcal{D}$ directions are parallel transported, and so J_{X_1} and J_{Y_1} along γ are precisely the parallel transports of their initial values. That is, $\nabla_{\dot{\gamma}} J_{X_1} = \nabla_{\dot{\gamma}} J_{Y_1} = 0$. Therefore,

$$\frac{d}{dt}\langle J_{X_1}(t), J_{Y_1}(t)\rangle = \langle \nabla_{\dot{\gamma}} J_{X_1}, J_{Y_1}\rangle + \langle J_{X_1}, \nabla_{\dot{\gamma}} J_{Y_1}\rangle = 0,$$

giving that $\langle J_{X_1}(t), J_{Y_1}(t)\rangle$ is constant in t. So,

$$\langle J_{X_1}(1), J_{Y_1}(1)\rangle = \langle J_{X_1}(0), J_{Y_1}(0)\rangle = \langle X_1, Y_1\rangle,$$

giving that $\bar{g}(X_1, Y_1) = \langle d\Phi(X_1, 0), d\Phi(Y_1, 0)\rangle_g$ as desired. Showing that the metric is preserved on $g_{L^\perp}$ follows by the same techniques. $\diamond$

Substituting in to the expansion yields

$$\begin{aligned}\langle d\Phi(X_1, X_2), d\Phi(Y_1, Y_2)\rangle_g &= \langle d\Phi(X_1, 0), d\Phi(Y_1, 0)\rangle_g + \langle d\Phi(0, X_2), d\Phi(0, Y_2)\rangle_g \\ &= \bar{g}(X_1, Y_1) + g_{L^\perp}(X_2, Y_2),\end{aligned}$$

and so $\Phi^* g = \bar{g} \oplus g_{L^\perp}$. □

So the manifold can be split locally. We want to use properties of the Busemann function g_+ to extend this to a global result; in particular, since ∇g_+ is parallel and nonvanishing, we can propagate the splitting globally by gluing the de Rham charts using a global coordinate along g_+ and a global transversal, which is a level set of g_+. We will do so by constructing an isometry F such that $F^*g = dt^2 \oplus h$. Before we present and prove the global version of de Rham splitting in the special case of ∇g_+, first consider the following technical lemma.

Lemma 5.6. $\nabla_{\partial_t} X(t) = \nabla_{X(t)}(\nabla g_+)$ where $F(t,p) = \varphi_t(p)$ is the flow of ∇g_+, $X \in T_pN$, and $X(t) = d\varphi_t(X)$.

Proof. Pick a curve $\sigma(s) \subset N$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Consider the variation $\Gamma(s,t) = \varphi_t(\sigma(s))$, and define $T = \partial_t\Gamma$ and $S = \partial_s\Gamma$. Then $T(s,t) = (\nabla g_+)(\Gamma(s,t))$ since φ_t is the flow of ∇g_+, and $S(0,t) = X(t)$.

Since the Levi-Civita connection is torsion free, $\nabla_T S - \nabla_S T = [T,S]$. But since S, T are coordinate vector fields from a smooth map Γ, the Lie bracket is 0, hence the derivatives commute. At $s = 0$, $T(0,t) = (\nabla g_+)(\varphi_t(p))$ and $S(0,t) = X(t)$, hence

$$\nabla_{\partial_t} X(t) = \nabla_T S|_{s=0} = \nabla_S T|_{s=0}.$$

To compute the right hand side, notice that $T(s,t) = (\nabla g_+)(\Gamma(s,t))$, and so differentiating in the S direction is just covariant differentiation of ∇g_+, that is, $\nabla_S T = \nabla_S(\nabla g_+)$. Evaluating at $s = 0$ gives $\nabla_{\partial_t} X(t) = \nabla_{X(t)}(\nabla g_+)$. $\square$

Theorem 5.7 (Globalized Local de Rham)**.** Suppose g_+ is a smooth forward Busemann function such that $|\nabla g_+| \equiv 1$ and $\Delta g_+ = 0$, that is, ∇g_+ is parallel. Then there exists an isometry F such that $F^* g = dt^2 \oplus h$ globally.

Proof. For notational simplicity, we write $V = \nabla g_+$, and define the level set $N := g_+^{-1}(0)$. Since $|\nabla g_+| = 1$, we have that 0 (in fact, any point) is a regular value. Hence the Regular Level Set Theorem (Corollary 5.14 in [Lee13]) gives that N is a smooth embedded hypersurface, and

$$T_pN = \{X \in T_pM : \langle V_p, X\rangle = 0\} = \ker(dg_+)_p = V_p^\perp.$$

Note too that we can split $T_pM = T_pN \oplus (T_pN)^\perp$; the right term is therefore $(T_pN)^\perp = (V_p^\perp)^\perp = \mathbb{R} \cdot V_p$ since $(v^\perp)^\perp = \operatorname{span}(v)$ for $v \neq 0$. With this, Theorem 5.2 gives that we can integrate $\mathbb{R} \cdot V_p \oplus T_pN$ to a local Riemannian product; we claim that since $V = \nabla g_+$ is parallel (from the Bochner formula) and nonvanishing, we can extend this Riemannian product to be global.

First, we show that the flow φ_t of V exists for all time, and allows us to move through the level sets linearly. By definition, we have $\frac{d}{dt}\varphi_t(x) = V_{\varphi_t(x)}$. Since $|V| = 1$ (that is, it is bounded) and M is complete, the flow exists for all $t \in \mathbb{R}$; that is, there is no blow-up and the flow lines are unit speed geodesics defined for all time. Then we compute

$$\frac{d}{dt} g_+(\varphi_t(x)) = dg_+(V) = \langle V, V\rangle = |V|^2 = 1,$$

hence $g_+(\varphi_t(x)) = g_+(x) + t$. In particular, if $x \in N$, then $g_+(x) = 0$, and so $g_+(\varphi_t(x)) = t$; that is, φ_t is a diffeomorphism of N onto the level set $g_+^{-1}(t)$.

Suppose we then define the map $F : \mathbb{R} \times N \to M$ by $F(t,p) = \varphi_t(p)$. Given any $x \in M$, take $t = g_+(x)$; then $p := \varphi_{-t}(x)$ is such that $g_+(p) = g_+(x) - t = 0$, and so $p \in N$ and $x = \varphi_t(p) = F(t,p)$, and so F is surjective. Also if $F(t,p) = F(s,q)$, then we apply g_+ to get that $t = g_+(F(t,p)) = g_+(F(s,q)) = s$, hence $s = t$, and also $\varphi_t(p) = \varphi_t(q)$ implies $p = q$ by the uniqueness of ODE flow, hence F is injective.

It remains to show that this bijection is an isometry; to do so, we show that dF is an isomorphism at all points, giving that F is a local diffeomorphism by the inverse function theorem, which is then a global diffeomorphism since it is a bijection.

Proof. We want to show that $dF : T_{(t,p)}(\mathbb{R} \times N) \to T_{F(t,p)}M$ is an isomorphism. Notice that $\dim(T_{(t,p)}(\mathbb{R} \times N) = 1 + (n-1) = n = \dim(T_{F(t,p)}M)$, and so it suffices to show that dF is injective. In particular, at $(t,p) \in \mathbb{R} \times N$, we can split $T_{(t,p)}(\mathbb{R} \times N) = \mathbb{R}\partial_t \oplus T_pN$, so if $a\partial_t + v \in \mathbb{R}\partial_t \oplus T_pN$, we want to show $dF(a\partial_t + v) = 0$ implies $a = 0$ and $v = 0$.

Notice that $dF_{(t,p)}(\partial_t) = \frac{\partial}{\partial t}F(t,p) = \frac{d}{dt}\varphi_t(p) = V(\varphi_t(p))$, hence $dF(\partial_t) = V$. Also, we claim that $dF(v) \in V^\perp$. Indeed, recall $N = g_+^{-1}(0)$ and $T_pN = \ker(dg_+)_p$, hence $d(g_+)_p(v) = 0$. By the chain rule, $d(g_+)_{F(t,p)}(dF(v)) = d(g_+ \circ F)_{(t,p)}(v)$, but $g_+(\varphi_t(p)) = (g_+ \circ F)(t,p) = g_+(p) + t$, and so $v(g_+ \circ F) = v(g_+(p)) = 0$, since $g_+(p) = 0$ on N.

Hence $d(g_+)(dF(v)) = 0$. But by definition of the gradient, $dg_+(Y) = \langle V, Y\rangle$, hence $\langle V, dF(v)\rangle = 0$, and so $dF(v) \in V^\perp$ as claimed.

Last, we claim that $T_{F(t,p)}M = \mathbb{R}V \oplus V^\perp$. For notational simplicity, take $x = F(t,p)$; then $V_x \neq 0$ since $|V| = 1$ and $V^\perp(x) = \{Y : \langle Y, V\rangle = 0\}$. By definition, this $V^\perp$ is a hypersurface. Since $\dim(\mathbb{R}V) = 1$ and $\dim(V^\perp) = n-1$, it remains to show $\mathbb{R}V \oplus V^\perp = \{0\}$. Indeed, suppose $w \in \mathbb{R}V_x \cap V_x^\perp$, then $w = \lambda V_x$ for some λ, and also $0 = \langle w, V_x\rangle = \lambda |V_x|^2$. Since $|V_x|^2 = 1$, we must have $\lambda = 0$, hence $w = 0$. Therefore, $T_{F(t,p)}M$ is in fact a direct sum.

So, suppose $dF(a\partial_t + v) = 0$, hence $aV + dF(v) = 0$. But $aV \in \mathbb{R}V$ and $dF(v) \in V^\perp$, and so we must have $aV = 0$ and $dF(v) = 0$ by the direct sum decomposition shown earlier. But, $|V| = 1$ so $a = 0$, and since $dF(v) = (\varphi_t)_* v$ where φ_t is a flow hence diffeomorphism, the map $(\varphi_t)_* : T_pM \to T_{F(t,p)}M$ is an isomorphism, hence $dF(v) = 0$ implies $v = 0$. So dF is injective, hence an isomorphism. ◇

As discussed, since dF is an isomorphism, F is a local diffeomorphism, and since F is also a bijection, F is a global diffeomorphism. We now want to show that it is an isometry; we will check this on pairs of vectors belonging to either $\mathbb{R}\partial_t$ or $V^\perp$. Take h to be a restriction of the metric g, so $h_p(X,Y) = g_p(X,Y)$ where $X, Y \in T_pN \subset T_pM$. If we give $\mathbb{R} \times N$ the metric $\tilde{g} = dt^2 \oplus h$, then $\tilde{g}(\partial_t, \partial_t) = 1$, $\tilde{g}(\partial_t, X) = 0$, and $\tilde{g}(X,Y) = h_p(X,Y)$; we want to show that $F^*g = \tilde{g}$. By definition of pullback, we show the first two conditions, where $X \in T_pN$:

$$(F^*g)_{(t,p)}(\partial_t, \partial_t) = g_{F(t,p)}(dF(\partial_t), dF(\partial_t)) = g_{F(t,p)}(V, V) = 1,$$
$$(F^*g)_{(t,p)}(\partial_t, X) = g_{F(t,p)}(dF(\partial_t), dF(X)) = g_{F(t,p)}(V, dF(X)) = 0,$$

where the last equality comes from the fact that $dF(X) \in V^\perp$, hence gives an inner product of 0. For the last condition, fix $p \in N$ and $X, Y \in T_pN$. If we set $f(t) = g(X(t), Y(t))$, where $X(t) = d\varphi_t(X)$ and $Y(t) = d\varphi_t(Y)$, we claim that $f'(t) = 0$.

Proof. See that $\frac{d}{dt} g(X(t), Y(t)) = g(\nabla_{\partial_t} X(t), Y(t)) + g(X(t), \nabla_{\partial_t} Y(t))$. We want to show that $\nabla_{\partial_t} X(t) = \nabla_{\partial_t} Y(t) = 0$. Lemma 5.6 gives that $\nabla_{\partial_t} X(t) = \nabla_{X(t)} V$, but $\nabla V = 0$ so $\nabla_{\partial_t} X(t) = 0$; similarly, we get $\nabla_{\partial_t} Y(t) = 0$. Hence $f'(t) = 0$. ◇

Therefore $f(t)$ is a constant function, and so $g(X(t), Y(t)) = g(X(0), Y(0)) = g(X, Y)$, which is equivalent to $h_p(X,Y)$ since $X, Y \in T_pN$. Therefore F is an isometry, and so we can extend the local de Rham splitting to a global splitting. □

6. Proof of Cheeger-Gromoll Splitting Theorem

Recall the titular theorem of this manuscript; we have now built up the techniques used to prove the Cheeger-Gromoll splitting theorem.

Theorem 6.1 ([CG71])**.** Let (M, g) be a complete Riemannian manifold of nonnegative Ricci curvature. Then (M, g) is isomorphic to the product $\mathbb{R}^k \times N$, where N contains no lines, and $\mathbb{R}^k$ has its usual flat metric.

Proof. If M contains no lines, then we are done. Otherwise, suppose that M contains a line $\gamma : \mathbb{R} \to M$. We denote the forward and backward Busemann functions associated to γ as g_+ and g_- respectively (Definition 3.14). Recall that Theorem 3.15 gives that g_+ is a weakly harmonic function, which we strengthen to g_+ being smooth and strongly harmonic via Corollary 3.18.

We apply the Bochner formula (Theorem 4.1) to g_+ and get

$$\frac{1}{2}\Delta|\nabla g_+|^2 = |\nabla^2 g_+|^2 + \langle \nabla\Delta g_+, \nabla g_+\rangle + \mathrm{Ric}(\nabla g_+, \nabla g_+).$$

But since g_+ is harmonic, $\langle \nabla\Delta g_+, \nabla g_+\rangle = \langle \nabla 0, \nabla g_+\rangle = 0$. Also, we showed that $|\nabla g_+| = 1$ in Proposition 3.13 and so the Bochner formula reduces to

$$0 = |\nabla^2 g_+|^2 + \mathrm{Ric}(\nabla g_+, \nabla g_+).$$

But we assumed that $\mathrm{Ric} \geq 0$, and surely $|\nabla^2 g_+|^2 \geq 0$, and so $|\nabla^2 g_+|^2 = \mathrm{Ric}(\nabla g_+, \nabla g_+) = 0$ as previously discussed in Section 4.

By definition, $\nabla^2 g_+(X, Y) = \langle \nabla_X(\nabla g_+), Y\rangle$, hence $0 = \langle \nabla_X(\nabla g_+), Y\rangle$ for vector fields X, Y. Since X, Y are arbitrary, $\nabla_X(\nabla g_+) = 0$ for all X, hence ∇g_+ is a parallel vector field. We can then define the rank-1 distribution $\mathcal{D} = \mathrm{span}(\nabla g_+)$. Recall that a distribution is said to be parallel if it is parallel transport invariant, or equivalently, if for any section Y of $\mathcal{D}$ and any vector field X, then $\nabla_X Y$ is a section of $\mathcal{D}$. Consider a section $Y = f\nabla g_+$ of $\mathcal{D}$. Then

$$\nabla_X Y = \nabla_X(f\nabla g_+) = (Xf)\nabla g_+ + f\nabla_X(\nabla g_+) = (Xf)\nabla g_+,$$

where in the second-to-last expression, the second term vanishes due to ∇g_+ being parallel. Hence $\mathcal{D}$ is a parallel distribution.

Since $\mathcal{D}$ is parallel, we claim that $\mathcal{D}^\perp$ is as well. Indeed, if Y is a section of $\mathcal{D}$ and Z is a section of $\mathcal{D}^\perp$, then $\langle Y, Z\rangle = 0$, and for any vector field X,

$$0 = X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Since $\mathcal{D}$ is parallel, $\nabla_X Y \in \mathcal{D}$, hence $\langle \nabla_X Y, Z\rangle = 0$, forcing $\langle Y, \nabla_X Z\rangle = 0$ for all $Y \in \Gamma(\mathcal{D})$, hence $\nabla_X Z \in \mathcal{D}^\perp$.

Lemma 5.5 gives that $\mathcal{D}$ implies the existence of a $\mathrm{Hol}(g)$-invariant subspace of the same dimension as the rank of $\mathcal{D}$, and Theorem 5.2 turns this into a local splitting. Theorem 5.7 then uses the flow of ∇g_+ to upgrade to a global splitting. By the constructions in the proofs of these splitting theorems, the Euclidean component dt^2 corresponds to the distribution $\mathcal{D}$ generated by the gradient of the Busemann function g_+, which is associated to the line γ. Therefore $(M, g) \cong (\mathbb{R} \times N, dt^2 \oplus g_N)$.

If M contains another independent line, that is, N contains a line, we repeat the same exact procedure on the manifold N to peel off another copy of $\mathbb{R}$. We continue doing so until N contains no lines, yielding the splitting $(M, g) \cong (\mathbb{R}^k \times N, \bar{g} \oplus g_N)$. □

7. Consequences

7.1. **Fundamental Group Structure.** Some particularly interesting consequences of the Cheeger-Gromoll splitting theorem are the restrictions on the topology that the product structure gives the manifold. Rather simply, consider first the following fact.

Proposition 7.1. If $M \cong \mathbb{R}^k \times N$, then $\pi_1(M, (x_0, y_0)) \cong \pi_1(N, y_0)$.

Proof. For a product manifold $M \times N$, we have $\pi_1(M \times N, (m_0, n_0)) \cong \pi_1(M, m_0) \times \pi_1(N, n_0)$. In our setting, $\mathbb{R}^k$ has trivial fundamental group, so $\pi_1(M, (x_0, y_0)) \cong \pi_1(N, y_0)$. $\square$

Beyond a simple reduction however, consider the case in which the universal cover $\widetilde{M}$ of the compact manifold M splits isometrically as $\mathbb{R}^k \times N$. As to be seen in Lemma 7.3, the group of deck transformations (which is isomorphic to $\pi_1(M)$ and will be notated as such throughout) acts by isometries and preserves the product structure. Restricting this action to the Euclidean factor defines a homomorphism into $\mathrm{Iso}(\mathbb{R}^k)$, and we will see that the image of this homomorphism is in fact discrete – equivalently, $\pi_1(M)$ is virtually abelian.

In order to show that $\pi_1(M)$ is virtually abelian when M is compact and $\widetilde{M}$ splits, we will prove a few lemmas about curvature and completeness of a manifold and its universal cover, as well as some properties about the action of deck transformations on $\widetilde{M}$. We also provide a critical fact in Lemma 7.5, which will guide our proof strategy.

Lemma 7.2. Let (M, g) be a Riemannian manifold, and let $\widetilde{M}$ be its universal cover with metric $\tilde{g} = \pi^* g$, with $\pi : \widetilde{M} \to M$ a Riemannian covering. Then $\mathrm{Ric}_M \geq 0$ if and only if $\mathrm{Ric}_{\widetilde{M}} \geq 0$, and M is complete if and only if $\widetilde{M}$ is complete.

Proof. Consider the Riemannian covering map $\pi : \widetilde{M} \to M$; this is a local isometry. Since local isometries preserve the metric, they preserve the curvature tensor:

$$d\pi(R^{\widetilde{M}}(X, Y)Z) = R^M(d\pi(X), d\pi(Y))(d\pi(Z)).$$

Therefore $\mathrm{Ric}_M \geq 0$ if and only if $\mathrm{Ric}_{\widetilde{M}} \geq 0$.

Next, suppose that M is complete. If $\tilde{\gamma} : (a, b) \to \widetilde{M}$ is a geodesic in $\widetilde{M}$, then $\gamma := \pi \circ \tilde{\gamma}$ is a geodesic in M. Since M is complete, γ exists for all time and extends to some geodesic $\gamma : \mathbb{R} \to M$. By the unique lifting property, there exists a unique lift of γ to $\widetilde{M}$, and this lift is defined for all $t \in \mathbb{R}$; hence $\widetilde{M}$ is complete.

Conversely, suppose $\widetilde{M}$ is complete and let $\gamma : (a, b) \to M$ be a geodesic in M. Then if $\tilde{p} \in \pi^{-1}(\gamma(c))$ (where $c \in (a, b)$), we can lift γ to a geodesic $\tilde{\gamma}$ in $\widetilde{M}$ such that $\tilde{\gamma}(c) = \tilde{p}$ and $\pi \circ \tilde{\gamma} = \gamma$. Since $\widetilde{M}$ is complete, $\tilde{\gamma}$ extends to all of $\mathbb{R}$, and so $\bar{\gamma} = \pi \circ \tilde{\gamma}$ is a geodesic in M defined for all $t \in \mathbb{R}$. $\square$

Lemma 7.3. Deck transformations are isometries and preserve the product $\widetilde{M} \cong \mathbb{R}^k \times N$.

Proof. First, notice that deck transformations are isometries: since a deck transformation τ satisfies $\pi \circ \tau = \pi$, we have $d\pi_{\tau(\tilde{p})} \circ d\tau_{\tilde{p}} = d\pi_{\tilde{p}}$. Since $\tilde{g} = \pi^* g$,

$$\tilde{g}_{\tau(\tilde{p})}(d\tau(X), d\tau(Y)) = g_{\pi(\tau(\tilde{p}))}(d\pi(d\tau(X)), d\pi(d\tau(Y)).$$

But, $\pi(\tau(\tilde{p})) = \pi(\tilde{p})$ by definition, hence $d\pi(d\tau(X)) = d\pi(X)$, and so $g_{\pi(\tilde{p})}(d\pi(X), d\pi(Y)) = \tilde{g}_{\tilde{p}}(X, Y)$, thus $\tau^* \tilde{g} = \tilde{g}$; therefore every deck transformation is an isometry $\tau : \widetilde{M} \to \widetilde{M}$.

Second, since τ is an isometry, $\tau_*(\nabla_X Y) = \nabla_{\tau_* X}(\tau_* Y)$. If V is a parallel vector field (i.e. $\nabla V = 0$), then for any X we have $\nabla_{\tau_* X}(\tau_* V) = \tau_*(\nabla_X V) = 0$, and so $\tau_* V$ is parallel. In proving the Cheeger-Gromoll theorem, we used the de Rham decomposition theorem, for which we constructed a distribution $\mathcal{D}$ where locally it is the span of a parallel vector field.

In particular, $\tau_*\mathcal{D} = \mathcal{D}$. Since τ is an isometry, $\tau_*(\mathcal{D}_{\tilde{p}}) = \mathcal{D}_{\tau(\tilde{p})}$, hence τ sends leaves of the $\mathcal{D}$-foliation to leaves of the $\mathcal{D}$-foliation, and the same for $\mathcal{D}^\perp$.

So, since $\mathcal{D}$ is tangent to the $\mathbb{R}^k$-factor, we want τ to send slices $\mathbb{R}^k \times \{y\}$ to $\mathbb{R}^k \times \{y'\}$, and each slice $\{x\} \times N$ to $\{x'\} \times N$. We must have that $\tau(x, y) = (Ax + b, \varphi(y))$, where $A \in O(k)$, $b \in \mathbb{R}^k$, and $\varphi \in \mathrm{Iso}(N)$. That is, deck transformations act as isometries and preserve the parallel distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ and their leaves. Hence they act by Euclidean isometries on $\mathbb{R}^k$ and by isometries of N on N. □

Lemma 7.4. Deck transformations act properly discontinuously on $\widetilde{M}$.

Proof. Recall that a group action (of deck transformations) is properly discontinuous if for every $\tilde{p} \in \widetilde{M}$, there exists a neighborhood $U \ni \tilde{p}$ such that $\tau(U) \cap U = \emptyset$ for all nonidentity $\tau \in \mathrm{Deck}(\widetilde{M}/M)$.

Since π is a covering map, there exists an open $V \subset M$ containing $p = \pi(\tilde{p})$ such that V is evenly covered. That is,

$$\pi^{-1}(V) = \bigsqcup_{\alpha \in A} U_\alpha,$$

where each U_α is open and $\pi|_{U_\alpha} : U_\alpha \to V$ is a diffeomorphism. Suppose U is the sheet of $\pi^{-1}(V)$ containing $\tilde{p}$. Then if τ is a nontrivial deck transformation, since $\pi \circ \tau = \pi$, the deck transformation τ must permute the sheets U_α. That is, $\tau(U) = U_\beta$ for some $\beta \in A$.

We claim that $\tau(U) \neq U$ if $\tau \neq e$. If not (i.e., $\tau(U) = U$), then $\tau|_U : U \to U$ is a diffeomorphism such that $\pi \circ \tau = \pi$. But $\pi|_U : U \to V$ is a diffeomorphism, so $\tau|_U = (\pi|_U)^{-1} \circ (\pi|_U) = \mathrm{id}_U$, but this would force $\tau = e$ since τ is a deck transformation that is identity on a nonempty open set, hence must be identity everywhere. Thus $\tau(U) \cap U = \emptyset$. □

Lemma 7.5. Let $1 \to G \to H \to K \to 1$ be an exact sequence with $K \cong H/G$. Then if G and K are virtually abelian, so is H.

Proof. Since K is virtually abelian, there exists a finite index abelian subgroup $K_0 \leq K$. Take $H_0 := \pi^{-1}(K_0)$, where $\pi : H \to K$ is the natural projection. Then H_0 has finite index in H since $[H : H_0] = [H : \pi^{-1}(K_0)] = [\pi(H) : K_0] = [K : K_0]$, and the sequence $1 \to G \to H_0 \to K_0 \to 1$ inherits exactness, which is easy to check. It suffices to show that H_0 is virtually abelian.

Since G is virtually abelian, let $G_0 \leq G$ be an abelian subgroup of finite index. Consider then the normalizer of G_0 in H_0,

$$H_1 = \{h \in H_0 : hG_0h^{-1} = G_0\}.$$

Since G_0 is finite index in G, only finitely many conjugates of G_0 occur, and so we must have that H_1 has finite index in H_0; also note that certainly $G_0 \trianglelefteq H_1$.

Consider then the conjugation action $\kappa : H_1 \to \mathrm{Aut}(G_0)$ given by $\kappa(h)(g) = hgh^{-1}$. Notice that $\ker \kappa = C_{H_1}(G_0) = \{h \in H_1 : hgh^{-1} = g \ \forall g \in G_0\}$. Since the conjugation action is a homomorphism (since conjugation by H_1 preserves G_0), and since G_0 has finite index in G, and since $G \trianglelefteq H$, conjugation sends G_0 to another subgroup of G of the same (finite) index. But, groups have only finitely many subgroups of a given finite index, and so there are only finitely many automorphisms of G_0 that can arise from conjugation by H_1. Therefore $|\kappa(H_1)| < \infty$. The first isomorphism theorem then gives that $[H_1 : \ker \kappa] < \infty$, hence the kernel has finite index in H_1, hence it has finite index in H.

Since the kernel of the conjugation map is the set of elements acting trivially on G_0, the elements of the kernel commute with all elements with G_0; hence $G_0 \subset Z(\ker \kappa)$. Hence we have a homomorphism $\pi|_{\ker \kappa} : \ker \kappa \to K_0$ with $\ker(\pi|_{\ker \kappa}) = G_0$. By the first isomorphism theorem, $(\ker \kappa)/G_0 \cong \pi(\ker \kappa)$, but $\pi(\ker \kappa) \leq K_0$, hence $(\ker \kappa)/G_0 \leq K_0$ up to isomorphism, hence it is abelian.

So, for $x, y \in \ker \kappa$, $[x, y] \in G_0 \subset Z(\ker \kappa)$. Hence $\ker \kappa$ is nilpotent with central abelian group G_0. Then the subgroup $Q = \langle [x, y] : x, y \in \ker \kappa \rangle \leq \ker \kappa$ is finitely generated and abelian, and so $R := \{h \in \ker \kappa : h \equiv 0 \bmod Q\}$ is abelian with finite index in $\ker \kappa$. Hence R is abelian with finite index in H; H is virtually abelian. □

We are now ready to prove the following theorem about the fundamental group of M. Note that the proof is not entirely self contained; there are sub-proofs listed as lemmas immediately following the proof of this theorem. The goal of these claims is to establish the hypothesis of Lemma 7.5, which will then imply the desired result.

Theorem 7.6. If the universal cover $\widetilde{M}$ of a compact manifold M splits in accordance with Cheeger-Gromoll, then $\pi_1(M)$ is virtually abelian.

Proof. Since all finite groups contain an abelian subgroup, necessarily of finite index, finite groups are virtually abelian. So, let us restrict our attention to the case $|\pi_1(M)| = \infty$. This implies the existence of nontrivial deck transformations of the cover $\widetilde{M}$, since there is a canonical isomorphism $\mathrm{Deck}(\widetilde{M}/M) \cong \pi_1(M)$, which is therefore of infinite order.

Suppose $\tau \in \pi_1(M)$ is a deck transformation acting as an isometry. Then $\tau(x, y) = (Ax + b, \varphi(y))$, where $A \in O(k)$, $b \in \mathbb{R}^k$, and $\varphi \in \mathrm{Iso}(N)$ (see Lemma 7.3). Let us define the map $\rho : \pi_1(M) \to \mathrm{Iso}(\mathbb{R}^k)$ by $\rho(\tau)(x) = Ax + b$.

We claim that this map ρ is a homomorphism, that is $\rho(\tau_1 \circ \tau_2) = \rho(\tau_1) \circ \rho(\tau_2)$. Let $\tau_1(x, y) = (A_1 x + b_1, \varphi_1(y))$ and $\tau_2(x, y) = (A_2 x + b_2, \varphi_2(y))$. Then

$$(\tau_1 \circ \tau_2)(x, y) = \tau_1(A_2 x + b_2, \varphi_2(y)) = (A_1 A_2 x + A_1 b_2 + b_1, \varphi_1(\varphi_2(y))),$$

hence $\rho(\tau_1 \circ \tau_2) = A_1 A_2 x + A_1 b_2 + b_1$. Likewise,

$$(\rho(\tau_1) \circ \rho(\tau_2))(x) = \rho(\tau_1)(A_2 x + b_2) = A_1(A_2 x + b_2) + b_1 = A_1 A_2 x + A_1 b_2 + b_1.$$

Hence $\rho(\tau_1 \circ \tau_2) = \rho(\tau_1) \circ \rho(\tau_2)$, so ρ is a homomorphism. Thus the image $\rho(\pi_1(M))$ is a subgroup of $\mathrm{Iso}(\mathbb{R}^k)$.

Since ρ is a homomorphism, we can write the short exact sequence

$$0 \to \ker \rho \to \pi_1(M) \to \rho(\pi_1(M)) \to 0.$$

By Lemma 7.5, in order to show that $\pi_1(M)$ is virtually abelian, it suffices to show that both $\ker \rho$ and $\rho(\pi_1(M))$ are virtually abelian. In order to show this, we establish four claims, which will be proven as separate lemmas immediately following this proof for cleanliness – once these have been done, we have the result.

Claim 1: $\rho(\pi_1(M))$ acts cocompactly on $\mathbb{R}^k$, and $\ker\rho$ acts cocompactly on N (Lemma 7.7).
Claim 2: $\rho(\pi_1(M))$ is a discrete subgroup of $\mathrm{Iso}(R^k)$ (Lemma 7.8).
Claim 3: $\rho(\pi_1(M))$ is virtually abelian (Lemma 7.9).
Claim 4: $\ker(\rho)$ is finite, hence virtually abelian (Lemma 7.10). ☐

Lemma 7.7 (Claim 1)**.** $\rho(\pi_1(M))$ acts cocompactly on $\mathbb{R}^k$, and $\ker\rho$ acts cocompactly on N.

Proof. First fix $y_0 \in N$ and consider $S := \mathbb{R}^k \times \{y_0\} \subset \widetilde{M}$. Since the total action on $\widetilde{M}$ is cocompact, there exists a compact fundamental domain $D \subset \mathbb{R}^k \times N$ such that $\pi_1(M) \cdot D = \mathbb{R}^k \times N$. Then we claim the projection $\pi_{\mathbb{R}^k}(D)$ on to the $\mathbb{R}^k$ factor is compact. Take some $x \in \mathbb{R}^k$; then $(x, y_0) \in \widetilde{M}$ and so there exists some $\tau \in \pi_1(M)$ such that $\tau^{-1}(x, y_0) = (\rho(\tau^{-1})x, \varphi_{\tau^{-1}}(y_0)) \in D$. Projecting to $\mathbb{R}^k$, we have $\rho(\tau^{-1})x \in \pi_{\mathbb{R}^k}(D)$, hence $x \in \rho(\tau)(\pi_{\mathbb{R}^k}(D))$ since ρ is a homomorphism. Hence every $x \in \mathbb{R}^k$ lies in the $\rho(\pi_1(M))$ orbit of the compact set $\pi_{\mathbb{R}^k}(D)$, and so $\mathbb{R}^k/\rho(\pi_1(M))$ is compact; that is, $\rho(\pi_1(M))$ acts cocompactly on $\mathbb{R}^k$.

Second, since $\pi_1(M)$ acts cocompactly on $\mathbb{R}^k \times N$, we claim the projection on to the N factor $C := \pi_N(D)$ is compact. Take any $x \in \mathbb{R}^n$ and fix $y \in N$; since $\pi_1(M) \cdot D = \widetilde{M}$, there exists $\tau \in \pi_1(M)$ such that $\tau^{-1}(x, y) \in D$. We can write $\tau^{-1}(x, y) = (\rho(\tau^{-1})x, \varphi_{\tau^{-1}}(y))$. Then $\pi_N(\tau^{-1}(x, y)) = \varphi_{\tau^{-1}}(y) \in C$, hence $y \in \varphi_\tau(C)$ (φ_τ is an isometry, hence bijection, so inverses are defined; $\varphi_{\tau^{-1}} = (\varphi_\tau)^{-1}$ because $\varphi : \pi_1(M) \to \mathrm{Iso}(N)$ is a homomorphism, following from Lemma 7.3). So, N is covered by the deck transformations of the compact set C.

But, if τ_1, τ_2 have the same image under ρ, then they differ by an element of K. Since $\rho(\pi_1(M))$ acts on $\mathbb{R}^k$ by Euclidean isometries, the only remaining freedom in the N direction must come from $\ker\rho$. In particular, if we fix a compact fundamental domain $\Xi \subset \mathbb{R}^k$ for the cocompact action of $\rho(\pi_1(M))$ on $\mathbb{R}^k$, then any (x, y) can first be moved by some $\tau \in \pi_1(M)$ so that its $\mathbb{R}^k$ coordinate lies in Ξ, and then after this, any further deck transformation preserving the $\mathbb{R}^k$ component must lie in K. Projecting to N, we have then that $N = K \cdot C'$ for some compact $C' \subset N$, hence N/K is compact; that is K acts cocompactly on N. ☐

Lemma 7.8 (Claim 2)**.** $\rho(\pi_1(M))$ is a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^k)$.

Proof. Let $\tilde{p} \in \widetilde{M}$, and pick some $U \ni \tilde{p}$ such that $\tau(U) \cap U = \emptyset$ for all nonidentity $\tau \in \mathrm{Deck}(\widetilde{M}/M)$. Since $\widetilde{M}$ is a metric space and U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(\tilde{p}) \subset U$. Then for $\tau \neq e$, $\tau(B_\varepsilon(\tilde{p})) \subset \tau(U)$, and also $\tau(B_\varepsilon(\tilde{p})) \cap B_\varepsilon(\tilde{p}) = \emptyset$. In particular, $\tau(\tilde{p}) \notin B_\varepsilon(\tilde{p})$, and so $d(\tilde{p}, \tau(\tilde{p})) \geq \varepsilon$.

Now consider $K = \ker\rho \leq \pi_1(M)$. Then $\ker\rho$ consists of deck transformations that act trivially on $\mathbb{R}^k$, and so they are of the form $\kappa(x, y) = (x, \varphi_\kappa(y))$. Since $M \cong \widetilde{M}/\pi_1(M)$ (by definition of universal cover) is compact, $\pi_1(M)$ acts cocompactly on $\widetilde{M}$ by deck transformations. By Lemma 7.7, $\rho(\pi_1(M))$ acts cocompactly on $\mathbb{R}^k$, and K acts cocompactly on N.

Let $C \subset N$ be a compact set such that $K \cdot C = N$ by cocompactness. Pick a compact ball $\overline{B_r(x_0)} \subset \mathbb{R}^k$ for any $r > 0$, and set $Q := \overline{B_r(x_0)} \times C \subset \widetilde{M}$; this is a compact set, being a product of compact sets.

Now, let W be a neighborhood of id in $\text{Iso}(\mathbb{R}^k)$ that is small enough so that for all $\eta \in W$,

$$\eta(\overline{B_r(x_0)}) \cap \overline{B_r(x_0)} \neq \emptyset.$$

We can do this since if $|\eta(x_0) - x_0| < r$ then the two balls $B_r(x_0)$ and $B_r(\eta(x_0))$ must intersect; but, $\eta(B_r(x_0)) = B_r(\eta(x_0))$ since η is an isometry, hence $\eta(B_r(x_0)) \cap B_r(x_0) \neq \emptyset$, and taking closures gives this result. Take now $\tau \in \pi_1(M)$ such that $\rho(\tau) \in W$. By the previous display style equation, pick $x \in \overline{B_r(x_0)} \cap \rho(\tau)(\overline{B_r(x_0)})$. Since K acts cocompactly on N, the point $\varphi_\tau(y_0) \in N$ (the image of the isometry on N determined by τ) can be moved back in to C by some $\kappa \in K$. In particular, take κ such that $\varphi_\kappa(\varphi_\tau(y_0)) \in C$. Now $\rho(\kappa) = \text{id}$ since $\kappa \in \ker \rho$, hence $\rho(\kappa\tau) = \rho(\tau)$ since ρ is a homomorphism.

If we consider the point $(x, y_0) \in \overline{B_r(x_0)} \times N$, then under the action of $\kappa\tau$,

$$(\kappa\tau)(x, y_0) = (\rho(\tau)(x), \varphi_\kappa(\varphi_\tau(y_0))).$$

Since $\rho(\tau)(x) \in \overline{B_r(x_0)}$ and since $\varphi_\kappa(\varphi_\tau(y_0)) \in C$, we have $(\kappa\tau)(x, y_0) \in Q$. But, this means that if we take any $(x, y) \in Q \subset \overline{B_r(x_0)} \times N$, then its image under $\kappa\tau$ also lands in Q, and so $(\kappa\tau)(Q) \cap Q \neq \emptyset$ for every $\tau \in \pi_1(M)$ such that $\rho(\tau) \in W \subset \text{Iso}(\mathbb{R}^k)$.

Finally, since $\pi_1(M)$ acts properly discontinuously on $\widetilde{M}$, the set $\{\tau \in \pi_1(M) : \tau(Q) \cap Q \neq \emptyset\}$ is finite for any compact Q in $\widetilde{M}$. Applying this to the Q we constructed gives that $S := \{\tau \in \pi_1(M) : \tau(Q) \cap Q \neq \emptyset\}$ is a finite set. But since $(\kappa\tau)(Q) \cap Q \neq \emptyset$, if $\rho(\tau) \in W$ then $\kappa\tau \in S$ for some $\kappa \in K$. But $\rho(\kappa\tau) = \rho(\tau)$, and so $\rho(\kappa\tau) = \rho(\tau) \in \rho(S)$, hence $\rho(\pi_1(M)) \cap W \subset \rho(S)$.

Since S is finite, so is $\rho(S)$. If we shrink W arbitrarily small so that it contains no non-identity element of $\rho(S)$, then $\rho(\pi_1(M)) \cap W = \text{id}$, hence $\rho(\pi_1(M))$ is discrete. ☐

Lemma 7.9 (Claim 3)**.** $\rho(\pi_1(M))$ is virtually abelian.

Proof. The structure of the Euclidean isomorphism group is $\text{Iso}(\mathbb{R}^k) = \mathbb{R}^k \rtimes O(k)$, and an element is (A, b) acting by $x \mapsto Ax + b$. Consider the linear part $A \in O(k)$. Define the homomorphism $\ell : \rho(\pi_1(M)) \to O(k)$ by $(A, b) \mapsto A$; then since $\rho(\pi_1(M))$ is discrete as shown in Lemma 7.8 and $O(k)$ is a compact (Lie) group, any discrete subgroup of a compact group is finite. Hence $\ell(\rho(\pi_1(M)))$ is finite, and so $|\ell(\rho(\pi_1(M)))| = [\rho(\pi_1(M)) : \ker \ell] < \infty$. Notice that $\ker \ell = \{(I, b) \in \rho(\pi_1(M))\}$, which is an additive subgroup of $\mathbb{R}^k$. Since $\rho(\pi_1(M))$ is discrete in $\text{Iso}(\mathbb{R}^k)$, $\ker \ell$ is a discrete subgroup of $\mathbb{R}^k$, and therefore $\ker \ell \cong \mathbb{Z}^r$ for $r \leq k$ by some linear algebra. So, $[\rho(\pi_1(M)) : \ker \ell] < \infty$ and $\rho(\pi_1(M))$ is virtually free abelian. ☐

Lemma 7.10 (Claim 4)**.** The group $\ker \rho$ is finite (hence virtually abelian).

Proof. Recall that we are in the setting that M is compact with universal cover $\widetilde{M} \cong \mathbb{R}^k \times N$, and we have shown that $K = \ker \rho$ acts cocompactly on N. Since $K \leq \pi_1(M)$, each $\kappa \in K$ is a deck transformation of $\widetilde{M}$, hence it acts freely on $\widetilde{M}$. By definition of the kernel, each $\kappa \in K$ fixes the Euclidean factor, so $\kappa(x, y) = (x, \varphi_\kappa(y))$ for some $\varphi_\kappa \in \text{Iso}(N)$; hence K acts freely on N. Further, since deck transformations act properly, the action of K also is proper on $\widetilde{M}$, hence it is proper on N. Since N/K is compact, and since the action of K is smooth, free, and proper, we can apply Theorem 21.13 [Lee13] to get that $\pi : N \to N/K$ is a smooth covering map.

Since N/K is compact, we can choose finitely many evenly covered open sets $U_1, ..., U_m \subset N/K$ covering N/K, with each $\overline{U_i}$ compact. For each i, take a sheet $V_i \subset \pi^{-1}(U_i)$; then $V = \bigcup_1^m \overline{V_i}$ is compact. Since every point in N/K lies in some U_i, every K-orbit in N meets V, thus $K \cdot V = N$.

Now, we can apply Lemma 21.11 of [Lee13] to the proper action of K on N. For each $p \in V$, there exists an open neighborhood U_p such that $\kappa(U_p) \cap U_p = \emptyset$ for all nonidentity $\kappa \in K$. Since V is compact, we can take a finite collection of neighborhoods $\{U_{p_1}, ..., U_{p_k}\}$ covering V. We claim then that the set $A := \{\kappa \in K : \kappa(V) \cap V \neq \emptyset\}$ is finite. Indeed, if $\kappa(V) \cap V \neq \emptyset$, then for some i, j, we have $\kappa(U_{p_i}) \cap U_{p_j} \neq \emptyset$. For each fixed pair (i, j), properness of the action implies that only finitely many κ can satisfy this condition. Since there are only finitely many pairs (i, j) (there are k^2 to be precise), A is finite.

Finally, we claim that $A = K$. Let $\kappa \in K$. Since $K \cdot V = N$, the point $\kappa(p)$ lies in $\eta(V)$ for some $\eta \in K$, where $p \in V$. Then $\eta^{-1}\kappa(p) \in V$, hence $\eta^{-1}\kappa(V) \cap V \neq \emptyset$, hence $\eta^{-1}\kappa \in A$. Therefore every $\kappa \in K$ lies in a left translate of the set A, but because V meets every orbit exactly once after shrinking the chosen V_i if necessary, that translate must be unique, hence we can identify K with A. Therefore $\ker \rho$ is finite. □

Since we have shown Lemmas 7.7 through 7.10, we have proven Theorem 7.6. Further, we have the following corollary of Lemma 7.7 and Lemma 7.10.

Corollary 7.11. Let M be a compact manifold with Ric ≥ 0, and suppose that its universal cover $\widetilde{M}$ splits isometrically according to Theorem 6.1 as $\widetilde{M} \cong \mathbb{R}^k \times N$. Then N is compact.

Proof. Let $\pi_1(M)$ denote the group of deck transformations of $\widetilde{M}$. As we have seen, under the splitting $\widetilde{M} \cong \mathbb{R}^k \times N$, each deck transformation induces an isometry of the Euclidean factor, giving a homomorphism $\rho : \pi_1(M) \to \mathrm{Iso}(\mathbb{R}^k)$. By Lemma 7.7, $\ker \rho$ acts cocompactly on N, and so there exists a compact set $C \subset N$ such that

$$N = (\ker \rho) \cdot C = \bigcup_{\kappa \in \ker \rho} \kappa(C).$$

Since each $\kappa \in \ker \rho$ acts on N by an isometry (in particular, a homeomorphism), each $\kappa(C)$ is compact. Also $\ker \rho$ is finite by Lemma 7.10, and therefore N is a finite union of compact sets, hence N is compact. □

7.2. **Manifolds with Ends.** Another consequence of the Cheeger-Gromoll theorem concerns the geometry of the manifold at infinity. In particular, if a complete manifold of nonnegative Ricci curvature has two "ends," then we can construct minimizing geodesic connecting points in these different ends, and then use a compactness result to pass to a limiting geodesic line. An application of Theorem 6.1 then gives that M splits off a Euclidean component.

Definition 7.12. If M is a complete, noncompact manifold, then the connected components of $M \setminus K$ (where K is an arbitrarily large compact set) are called the ends of M.

If a manifold has two ends, then for some compact set K, we have that $M \setminus K = E_1 \sqcup E_2 \sqcup$ (bounded components).

Lemma 7.13 (Arzelà-Ascoli for Geodesics)**.** Let M be a complete manifold, and suppose $\gamma_i : [-a_i, b_i] \to M$ are unit speed geodesics such that $a_i, b_i \to \infty$ and $\gamma_i(0)$ lies in a fixed compact set. Then there exists a subsequence converging locally uniformly to a geodesic $\gamma : \mathbb{R} \to M$.

Proof. First notice that images stay in a compact set locally. If we restrict to some interval $[-R, R]$, then for some $s \in [-R, R]$, we have $d(\gamma_i(s), \gamma_i(0)) \leq |s| \leq R$ since the curves are unit speed. Therefore $\gamma_i([-R, R]) \subset B_R(\gamma_i(0))$. Since $\gamma_i(0) \in K$ and K is compact, $B_R(K)$ is compact and $\gamma_i([-R, R]) \subset B_R(K)$. Hence all curves stay inside a fixed compact set on $[-R, R]$. Also since the γ_i are unit speed, $d(\gamma_i(s), \gamma_i(t)) \leq |s - t|$, hence the γ_i are 1-Lipschitz, hence $\{\gamma_i\}$ is equicontinuous.

We have established the hypotheses for Arzelà-Ascoli, and so for each fixed R, the geodesic segment $\gamma_i|_{[-R,R]}$ has a uniformly convergent subsequence. We want to upgrade from convergence on just $[-R, R]$ to any compact interval. To do so, consider $[-1, 1]$. Then $\gamma_i([-1, 1]) \subset B_1(K)$ as shown earlier, and the γ_i are equicontinuous. Hence Arzelà-Ascoli gives the existence of $\gamma_i^{(1)}$ that converges uniformly on $[-1, 1]$. We can repeat this on $[-2, 2]$ to get $\gamma_i^{(2)}$ that converges uniformly on $[-2, 2]$. We can continue inductively to get that in general, $\gamma_i^{(R)}$ converges uniformly on $[-R, R]$; also note that $\gamma_i^{(R)} \subset \cdots \subset \gamma_i^{(2)} \subset \gamma_i^{(1)}$.

From this, define a new sequence by taking the Rth element form the Rth subsequence, that is, $\tilde{\gamma}_R = \gamma_R^{(R)}$. Notice that for any fixed m, all terms $\tilde{\gamma}_R$ such that $R \geq m$ lie in the subsequence $\gamma_i^{(m)}$, and $\gamma_i^{(m)}$ converges uniformly on $[-m, m]$. Since this holds for any m, we have $\tilde{\gamma}_i \to \gamma$ locally uniformly on $\mathbb{R}$. Further, γ is a geodesic, since $\tilde{\gamma}_i(0) \in K$ and $|\tilde{\gamma}_i'(0)| = 1$, hence the initial data $(\tilde{\gamma}_i(0), \tilde{\gamma}_i'(0))$ lie in a compact subset of TM, so after passing to a subsequence, $\tilde{\gamma}_i(0) \to p$ and $\tilde{\gamma}_i'(0) \to v$. Then there exists a unique geodesic γ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, and by smooth dependence on initial conditions, $\tilde{\gamma}_i \to \gamma$ locally uniformly. $\square$

Theorem 7.14. If a manifold M has two ends, then there exists a line.

Proof. Fix $p \in K$ and take sequences $\{x_i\} \subset E_1$ and $\{y_i\} \subset E_2$ such that $d(p, x_i) \to \infty$ and $d(p, y_i) \to \infty$; that is, these sequences diverge to different ends. Take $\gamma_i : [0, \ell_i] \to M$ to be minimizing geodesics from x_i to y_i. Since the points x_i and y_i lie in different ends, any path from x_i to y_i must pass through the compact set K, hence each γ_i intersects K. Pick t_i so that $z_i := \gamma_i(t_i) \in K$.

We now want to recenter the geodesics. Define $\sigma_i(s) = \gamma_i(s + t_i)$, hence $\sigma_i(0) = z_i \in K$. The domain of σ_i is then $[-t_i, \ell_i - t_i]$. Since $x_i, y_i \to \infty$, we also have that $t_i \to \infty$ and $\ell_i - t_i \to \infty$, so this domain extends to infinity in both directions.

By construction, the z_i lie in the compact set K. So, we can pass to a subsequence z_{i_k} which converges to $z \in K$. By Lemma 7.13, the curves σ_i have a subsequence converging to a geodesic $\sigma : \mathbb{R} \to M$. Each of the σ_i is minimizing between any two of its points, since it comes from a minimizing geodesic. Thus for $s < t$, $d(\sigma_i(s), \sigma_i(t)) = t - s$, and taking limits gives $d(\sigma(s), \sigma(t)) = t - s$, hence σ is globally minimizing on every interval, and so σ is a line. $\square$

Corollary 7.15. If a manifold with $\text{Ric} \geq 0$ has two ends, then it must split isometrically as $M \cong \mathbb{R} \times N$ with the flat metric on the Euclidean component.

Proof. By the previous theorem, if a manifold has two ends, there exists a line $\gamma : \mathbb{R} \to M$. Since $\mathrm{Ric} \geq 0$, we can apply Cheeger-Gromoll to get the result. $\square$

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